Statistical Field Theory 3

Path Integrals and Fermions

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1 Introduction

The central object in quantum Statistical Mechanics is the partition function

$$Z = \operatorname{Tr} e^{-\beta H} = \operatorname{Tr} T^N \tag{1.1}$$

where H is the quantum Hamiltonian and T is the transfer matrix. Recall for the Ising model we found the Hamiltonian to be

$$H = \sum_{n=1}^{N} -\lambda\sigma_1(n) - \sigma_3(n)\sigma_3(n+1), \tag{1.2}$$

[Sha17]. If we consider the partition function with $s_0 = s_i$ and $s_N = s_f$ (for some initial and final spin) recall that

$$\langle s_N = s_f| \, T^N \, |s_0 = s_i\rangle \iff \langle s_f| \, U(N\Delta\tau) \, |s_i\rangle \tag{1.3}$$

corresponds to the matrix element of the propagator U for imaginary time $N\Delta\tau$ between the states $\langle s_f|$ and $|s_i\rangle$. It follows that

$$U(f, i; \tau) = \langle f| \, U(\tau) \, |i\rangle \tag{1.4}$$

describing how a state evolves from position i to position f through the time evolution operator is a more general object to study. This is what we shall now do for the Feynman path integral for a generic Hamiltonian of a point particle in 1-dimension. We shall then study the operator formalism of fermionic and Grassmann numbers, then combine these to derive the path integral for the free fermion. Finally, we will express our Hamiltonian (1.2) in fermions to give the action that will describe the Conformal Field Theory.

2 The Feynman Path Integral

Now we turn to a path integral that was first studied by Feynman in his treatment of a particle in one dimension. In this formalism we have to consider all possible paths from a point x_i to the final point x_f.

We will perform the calculation in the real-time case t and keep $\hbar$. As before with the Ising Model, the game plan is to chop up $U(t) = e^{-itH/\hbar}$ into a product of N factors of $U(t/N)$, insert the resolution of the identity $N-1$ times and take the limit $N \to \infty$. Let's assume our Hamiltonian is time independent and has the form[1]

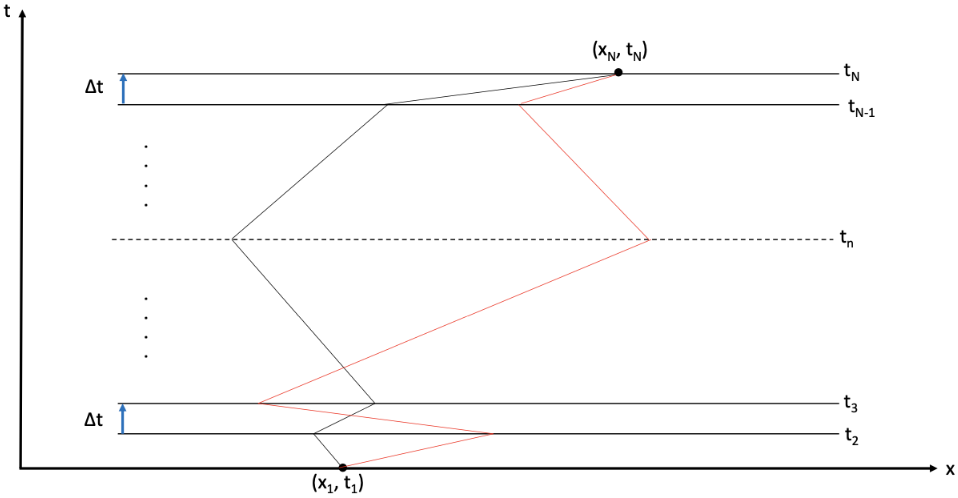
$$H = \frac{P^2}{2m} + V(X) \tag{2.1}$$

First, we may write the transfer matrix/unitary operator $U(t)$ as

$$U(t) = e^{-\frac{it}{\hbar}H} = \exp\left[-\frac{it}{\hbar}\left(\frac{P^2}{2m} + V(X)\right)\right] \tag{2.2}$$

Now we would like to be able to split the exponential, to justify this we make use of the following theorem.

[1] Note the relation to the SLT Hamiltonian with $m = 1$



Examples of two paths from (x_1, t_1) to (x_N, t_N). $\int dx_n \Rightarrow$ integrate over all paths $x_1 \to x_N$

Theorem 2.1 (Trotter Product Formula)**.** *Suppose that A and B are self-adjoint operators on $\mathbf{H}$, and that A and B is dense on $A+B$ and (essentially) self-adjoint on* $\mathrm{Dom}(A) \cap \mathrm{Dom}(B)$*. For all $\psi \in \mathbf{H}$, we have*

$$\lim_{N\to\infty} \left\| e^{it(A+B)}\psi - (e^{itA/N}e^{itB/N})^N \psi \right\| = 0, \tag{2.3}$$

meaning $(e^{itA/N}e^{itB/N})^N$ converges to $e^{it(A+B)}$ in the strong operator topology.

We will omit the proof (see [Hal13] Chapter 20) and not worry about what Hilbert space we are in but use this to justify splitting the exponential in the derivation of the path integral formula. Let $\varepsilon = t/N$. So we wish to compute,

$$\begin{aligned} \langle x'| U(t) |x\rangle &= \langle x'| \underbrace{U(t/N)\cdots U(t/N)}_{N \text{ times}} |x\rangle \\ &= \langle x'| e^{-\frac{i\varepsilon}{2m\hbar}P^2} \cdot e^{-\frac{i\varepsilon}{\hbar}V(X)} \cdots e^{-\frac{i\varepsilon}{2m\hbar}P^2} \cdot e^{-\frac{i\varepsilon}{\hbar}V(X)} |x\rangle \end{aligned} \tag{2.4}$$

The next step is to insert the resolution function the identity (**??**) between every two adjacent factors of $U(t/N)$. Setting $x' = x_n$ and $x = x_0$ and doing this gives us

$$\begin{aligned} \langle x_n| U(t) |x_0\rangle &= \langle x_n| e^{-\frac{i\varepsilon}{2m\hbar}P^2} e^{-\frac{i\varepsilon}{\hbar}V(X)} \left(\int_{-\infty}^{\infty} |x_{n-1}\rangle\langle x_{n-1}| dx_{n-1} \right) e^{-\frac{i\varepsilon}{2m\hbar}P^2} e^{-\frac{i\varepsilon}{\hbar}V(X)} \\ &\cdots \left(\int_{-\infty}^{\infty} |x_1\rangle\langle x_1| dx_1 \right) e^{-\frac{i\varepsilon}{2m\hbar}P^2} e^{-\frac{i\varepsilon}{\hbar}V(X)} |x_0\rangle \\ &= \int_{-\infty}^{\infty} \langle x_n| e^{-\frac{i\varepsilon}{2m\hbar}P^2} e^{-\frac{i\varepsilon}{\hbar}V(X)} |x_{n-1}\rangle\langle x_{n-1}| dx_{n-1} e^{-\frac{i\varepsilon}{2m\hbar}P^2} e^{-\frac{i\varepsilon}{\hbar}V(X)} \\ &\cdots \int_{-\infty}^{\infty} e^{-\frac{i\varepsilon}{2m\hbar}P^2} e^{-\frac{i\varepsilon}{\hbar}V(X)} |x_0\rangle \\ &= \int_{(\mathbb{R})^N} \langle x_n| e^{-\frac{i\varepsilon}{2m\hbar}P^2} e^{-\frac{i\varepsilon}{\hbar}V(X)} |x_{n-1}\rangle\langle x_{n-1}| \cdots |x_2\rangle\langle x_1| e^{-\frac{i\varepsilon}{2m\hbar}P^2} e^{-\frac{i\varepsilon}{\hbar}V(X)} |x_0\rangle \prod_{i=1}^{N} dx_i \\ &= \int_{(\mathbb{R})^N} \langle x_n| e^{-\frac{i\varepsilon}{2m\hbar}P^2} e^{-\frac{i\varepsilon}{\hbar}V(x_n)} |x_{n-1}\rangle\langle x_{n-1}| \cdots |x_2\rangle\langle x_1| e^{-\frac{i\varepsilon}{2m\hbar}P^2} e^{-\frac{i\varepsilon}{\hbar}V(x_0)} |x_0\rangle \prod_{i=1}^{N} dx_i \end{aligned} \tag{2.5}$$

where in the last step we acted on the ket's with the exponential of the position function $V(X)$ to get a number. Now in each case, we need to deal with the quantity

$$\langle x_n| e^{-\frac{i\varepsilon}{2m\hbar}P^2} e^{-\frac{i\varepsilon}{\hbar}V(x_{n-1})} |x_{n-1}\rangle = e^{-\frac{i\varepsilon}{\hbar}V(x_{n-1})} \langle x_n| e^{-\frac{i\varepsilon}{2m\hbar}P^2} |x_{n-1}\rangle \,. \tag{2.6}$$

So far, we only "know" that $X|x\rangle = x|x\rangle$ and $P|p\rangle = p|p\rangle$ as our operators act on our vectors. Let's take (2.6), ignoring the prefactor for now, and insert a complete set of momentum eigenstates to the left of the momentum operator

$$\begin{aligned}\langle x_n| e^{-\frac{i\varepsilon}{2m\hbar}P^2}|x_{n-1}\rangle &= \int_{-\infty}^{\infty} \langle x_n|p\rangle \langle p| e^{-\frac{i\varepsilon}{2m\hbar}P^2}|x_{n-1}\rangle \frac{dp}{2\pi\hbar} \\ &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{\frac{ix_n p}{\hbar}} e^{-\frac{i\varepsilon}{2m\hbar}p^2}\langle p|x_{n-1}\rangle\, dp \\ &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{\frac{ix_n p}{\hbar}} e^{-\frac{i\varepsilon}{2m\hbar}p^2} e^{-\frac{i}{\hbar}px_{n-1}}dp \\ &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \exp\left[\frac{i}{\hbar}\left(p(x_n - x_{n-1}) - \frac{p^2}{2m}\varepsilon\right)\right]\end{aligned} \tag{2.7}$$

Completing the square and using Fresnel's integral

$$\int_{-\infty}^{\infty} e^{-iap^2}dp = \sqrt{\frac{\pi}{ia}} \tag{2.8}$$

we finally [Vol21] have

$$\sqrt{\frac{m}{2\pi i\hbar\varepsilon}}\exp\left[\frac{im}{\hbar}\frac{(x_n - x_{n-1})^2}{\varepsilon}\right] \tag{2.9}$$

Therefore in summary each factor gives

$$\langle x_n| e^{-\frac{i\varepsilon}{2m\hbar}P^2} e^{-\frac{i\varepsilon}{\hbar}V(X)}|x_{n-1}\rangle = \sqrt{\frac{m}{2\pi i\hbar\varepsilon}}\exp\left[\frac{im}{\hbar}\frac{(x_n - x_{n-1})^2}{\varepsilon} - \frac{i\varepsilon}{\hbar}V(x_{n-1})\right]. \tag{2.10}$$

Subsituting into our propogator calculation (2.5) for each factor we obtain

$$\begin{aligned}\langle x_n| U(t)|x_0\rangle &= \lim_{N\to\infty}\int_{(\mathbb{R})^N}\prod_{n=1}^{N}\left(\sqrt{\frac{m}{2\pi i\hbar\varepsilon}}\exp\left[\frac{im}{\hbar}\frac{(x_n - x_{n-1})^2}{\varepsilon} - \frac{i\varepsilon}{\hbar}V(x_{n-1})\right]\right)\prod_{n=1}^{N}dx_n \\ &= \lim_{N\to\infty}\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{N/2}\int_{(\mathbb{R})^N}\exp\left[\sum_{n=1}^{N}\frac{im}{\hbar}\frac{(x_n - x_{n-1})^2}{\varepsilon} - \frac{i\varepsilon}{\hbar}V(x_{n-1})\right]\prod_{n=1}^{N}dx_n.\end{aligned} \tag{2.11}$$

As a quick aside, if you are unfamiliar with inserting complete sets of states to evaluate the result, one can invoke the following theorem at each point x_n instead in terms of the wave function itself.

Theorem 2.2. *Assuming that $\psi_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Then $\psi(x,t)$ that satisfies the Shroödinger equation may be computed for all $t \neq 0$ as*

$$\psi(x,t) = \sqrt{\frac{m}{2\pi it\hbar}}\int_{-\infty}^{\infty}\exp\left\{i\frac{m}{2t\hbar}(x-y)^2\right\}\psi_0(y)dy. \tag{2.12}$$

where $\psi_0(y) = \psi(y,0)$ is the initial condition.

We will not proof this here either (see [Hal13] Chapter 4) having given an alternative derivation in the conventional way.

Now emembering that $\varepsilon = t/N$ and assuming we can freely rearrange the order of integration, we obtain

$$\begin{aligned}&\langle x_n| U(t)|x_0\rangle \\ &= \lim_{N\to\infty} C\int_{(\mathbb{R})^N}\exp\left[\frac{i}{\hbar}\sum_{n=1}^{N}\varepsilon\left(\frac{m}{2}\left|\frac{x_n - x_{n-1}}{\varepsilon}\right|^2 - V(x_{n-1})\right)\right] \\ &\times dx_1 dx_2\cdots dx_N.\end{aligned} \tag{2.13}$$

where $C = \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{N/2}$. So far the argument is mostly rigorous, where our results can come from (2.1) and (2.2) and we assume we can freely exchange the order of integration. The non-rigorous part comes in attending to evaluate the limit. Let us think of the values x_n for $j = 0, \dots, N$ as constituting values of a path $x(s)$ at the points $s_n = n\varepsilon = nt/N$, so

$$x_n = x(nt/N). \tag{2.14}$$

Since the distance between s_{j-1} and s_j is ε, the term $\frac{x_n - x_{n-1}}{\varepsilon}$ is an approximation to the derivative of $x(s)$ with respect to s. Meanwhile, the sum over j in the right hand side of the exponent is an approximation of an integral. Thus is we then take the limit, in a totally nonrigourous fashion we obtain

$$\begin{aligned}&\langle x_n | \, U(t) \, | x_0 \rangle \\ &= C \int_{x(0)=x_0} \exp\left[\frac{i}{\hbar}\int_0^t \left(\frac{m}{2}\left|\frac{dx}{ds}\right|^2 - V(x(s))\right) ds\right] \mathcal{D}x\end{aligned} \tag{2.15}$$

where in integration is over all paths where $x(0) = x_0$, C is a normalisation constant, and $\mathcal{D}x$ is something like "Lebesgue measure" on all the space of paths $x(-)$ mapping $[0,t]$ into $\mathbf{R}$. The quantity x in the expression $\mathcal{D}x$ is a *path*, not a point in $\mathbb{R}$. The expression in the integral of the exponential is the *Lagrangian*, and the integral over it is called the *action*. Furthermore, the absolute value of the constant C is easily seen to be infinite, so we cannot take the right-hand side of this formula literally.r While this was all done in one-dimension, see [Hal13] for a derivation of a particle in N dimensions.

Now in trying to give rigorous meaning to the path integral formula of Feynman, Kac proceeded by considering the "imaginary time" time-evolution operator $\exp(-\tau H/\hbar)$ where $\tau = -it$. The original idea being that if one can use path integrals to understand the operator for τ, one can understand the "real time" operator by analytic continuation. One can make this rigorous by doing this Wick rotation, and defining

$$\mu = C \exp\left\{-\frac{1}{\hbar}\int_0^t \frac{m}{2}\left|\frac{dx}{ds}\right|^2 ds\right\} \mathcal{D}x \tag{2.16}$$

as *Weiner Measure* leading to the Feynman-Kac Formula, which we will not go into here, see [Hal13]. We need to breifly remark about the path integral approach to quantum field theory. We consider quantum field theories usually to be defined on space-time of dimension d, so that space has $d-1$ dimensions. The configuration space for the classica version of this theory is the collection of "spatial fields", that is maps $\phi(\mathbf{x})$ of $\mathbb{R}^{d-1}$ into some finite-dimensional vector space V. A *path* in ths space of fields is then a map $\phi(\mathbf{x}, t)$ of $\mathbb{R}^{d-1} \times \mathbb{R} \cong \mathbb{R}^d$ into V. A simple is called ϕ^4 theorym with is a Euclidean path integral

$$\int \exp\left\{-\frac{1}{\hbar}\int_{\mathbb{R}^d}\left[c_1 \left\|\nabla\phi(\mathbf{x})\right\|^2 + c_2\phi(\mathbf{x})^2 + c_4\phi(\mathbf{x})^4\right] d\mathbf{x}\right\} \tag{2.17}$$

Note that the action is integrated over spacetime, not just time. We will now consider ϕ^4 theory in these talks.

3 Fermion Operator Formalism and Coherent States

Now we would like to take Feynmann's formalism for a path integral interpretation and apply it to our 2D Ising model. To do this we're going to learn about the path integral for fermions, and then later argue that the operators of our Hamiltonian can be expressed as fermions, so are described by the same theory. For the moment we can think of our fermions Ψ as a change of variables from our spin operators σ. Before jumping into path integrals for fermions, lets get used to the operator formalism of fermions.

Fermions obey the anti-commutation relations

$$\begin{aligned}\{\Psi^\dagger, \Psi\} &= \Psi^\dagger\Psi + \Psi\Psi^\dagger = 1 \\ \{\Psi, \Psi\} &= \{\Psi^\dagger, \Psi^\dagger\} = 0\end{aligned} \tag{3.1}$$

The second equation tells us that

$$\Psi^2 = (\Psi^\dagger)^2 = 0. \tag{3.2}$$

Introducing the *number operator* $N = \Psi^\dagger\Psi$. We see that

$$\begin{aligned} N^2 &= \Psi^\dagger\Psi\Psi^\dagger\Psi \\ &= \Psi^\dagger(\Psi^\dagger\Psi + \Psi\Psi^\dagger - \Psi^\dagger\Psi)\Psi \\ &= \Psi^\dagger(\{\Psi^\dagger, \Psi\} - \Psi^\dagger\Psi)\Psi \\ &= \Psi^\dagger(1 - \Psi^\dagger\Psi)\Psi \\ &= N - \Psi^2(\Psi^\dagger)^2 \\ &= N. \end{aligned} \tag{3.3}$$

Therefore since $N(N-1) = 0$ this tells us that the Eigenvalues of the operator N must be 0 or 1 with normalised Eigenstates

$$N|0\rangle = 0|0\rangle = 0, \quad N|1\rangle = 1|1\rangle = |1\rangle\,. \tag{3.4}$$

Now to show that $\Psi^\dagger|0\rangle = |1\rangle$ (so $\Psi^\dagger$ is a creation operators) and $\Psi|1\rangle = |0\rangle$ (so Ψ is an annhilation operator). To show the first consider

$$N\Psi^\dagger|0\rangle = \Psi^\dagger\Psi\Psi^\dagger|0\rangle = \Psi^\dagger(1 - \Psi^\dagger\Psi)|0\rangle = \Psi^\dagger|0\rangle \tag{3.5}$$

which shows that N acting on $\Psi^\dagger|0\rangle$ has $N = 1$. So we must have $\Psi^\dagger|0\rangle = 1|1\rangle = |1\rangle$. This vector will have norm

$$\left\|\Psi^\dagger|0\rangle\right\|^2 = \langle 0|\Psi\Psi^\dagger|0\rangle = \langle 0|(1 - \Psi^\dagger\Psi)|0\rangle = \langle 0|0\rangle - N|0\rangle = 1. \tag{3.6}$$

Similarly one can show $\Psi|1\rangle = |0\rangle$. There are no other vectors in the Hilbert space, any attempts to produce more states is ruined by (3.2). Therefore the Pauli principle rules out more vectors, the states are either empty of singly occupied.

Now in order to evalate the path integral, we're going to need to have a resolution of the identity (identity operator we insert to derive the path integral). We will use fermion coherent states $|\psi\rangle$, which are the eigenstates of the annihilation operator

$$\Psi|\psi\rangle = \psi|\psi\rangle\,. \tag{3.7}$$

The eigenvalues ψ is a strange object, since if we act once more with Ψ, we see that $\psi^2 = 0$ since $\Psi^2 = 0$. Any ordinary variable that squares to zero is zero, but this not, it is a *Grassman variable*. One should think of Grassman numbers as elements of an exterior algebra of our vector (Hilbert space) where multiplication is the wedge product $\psi_1\psi_2 = \psi_1 \wedge \psi_2$. Other examples of wedge products is the determinant in $\mathbb{R}^2$ and the cross product in $\mathbb{R}^3$.

The wedge product $\wedge$ on $V = \Lambda(\mathcal{H})$ (the exterior algebra of our Hilbert space $\mathcal{H}$) which obey bilinearity, associativity and antisymmetry. Since we'll be in two dimensions the only property relevant to us is antisymmetry $\psi_1 \wedge \psi_2 = -\psi_2 \wedge \psi_1$ and $\psi \wedge \psi = 0$. We will not explicity write the wedge product, so $\psi^2 = \psi \wedge \psi$, but keep in mind that the Grassman variables obey these rules.

These variables anticommute with each other and with all fermionic creation and annihilation operators (they will therefore commute with a string containing an even number of such operators). Now, ψ does not commute with all the state vectors. If we suppose our Grassman numbers commute with the ground state $\psi|0\rangle = |0\rangle\psi$, then it follows that

$$\begin{aligned} \psi|1\rangle &= \psi\Psi^\dagger|0\rangle \\ &= (-\Psi^\dagger\psi + \{\psi, \Psi^\dagger\})|0\rangle \\ &= -\Psi^\dagger|0\rangle\psi + \{\psi, \Psi^\dagger\}|0\rangle \\ &= -|1\rangle\psi. \end{aligned} \tag{3.8}$$

Anti-commuting variables square to zero, Grassman variable anticommute with the creation and annihilation operators by definition the anti-commutator gives zero. We will propose our state that satsifies $\Psi|\psi\rangle = \psi|\psi\rangle$ can be expressed as

$$|\psi\rangle = |0\rangle - \psi|1\rangle\,, \tag{3.9}$$

where ψ is a grassman number. To check this

$$\begin{aligned} \Psi \left|\psi\right\rangle &= \Psi \left|0\right\rangle - \Psi\psi \left|1\right\rangle \\ &= 0 + \psi\Psi \left|1\right\rangle \\ &= \psi \left|0\right\rangle \\ &= \psi(\left|0\right\rangle - \psi \left|1\right\rangle) \\ &= \psi \left|\psi\right\rangle \end{aligned} \tag{3.10}$$

since $\psi^2 = 0$, as required. If we act on both sides of (3.10), with Ψ, the left and ride side vanish. It may be similarly verified that

$$\langle\overline{\psi}| \Psi^\dagger = \langle\overline{\psi}| \overline{\psi} \tag{3.11}$$

where

$$\langle\overline{\psi}| \Psi^\dagger = \langle 0| - \langle 1| \overline{\psi} = \langle 0| + \overline{\psi} \langle 1|. \tag{3.12}$$

First note that the coherent state vectors are not the usual vectors from a complex vector space since they are linear combinations with grassman coefficients. Second note that $\overline{\psi}$ is not the complex conjugate of ψ, and $\langle\overline{\psi}|$ is not the adjoint of $|\psi\rangle$. We can therefore see change of grassman variables where ψ and $\overline{\psi}$ undergo unrelated transformations. Sometimes $\overline{\psi}$ is denoted as η to emphasise the difference. We will call it $\overline{\psi}$ to remind us that in a theory where every operator Ψ has an adjoint $\Psi^\dagger$, for every label ψ there is another independent label $\overline{\psi}$. The inner product of two coherent states is

$$\begin{aligned} \langle\overline{\psi}|\psi\rangle &= \big(\langle 0| - \langle 1| \overline{\psi}\big) \left(\left|0\right\rangle - \psi \left|1\right\rangle\right) \\ &= \langle 0|0\rangle + \langle 1| \overline{\psi}\psi \left|0\right\rangle \\ &= 1 + \overline{\psi}\psi \\ &= e^{\overline{\psi}\psi}, \end{aligned} \tag{3.13}$$

since $(\overline{\psi}\psi)^2 = 0$. Any function of Grassmann variables can be expanded as follows

$$F(\psi) = F_0 + F_1\psi \tag{3.14}$$

where no higher powers are possible.

Before we can do the path integral we have to learn how to integrate over Grassmann numbers. We will now *define* integrals over Grassmann numbers. These have no geometric significance (as areas or volumes) are formally defined. We just have to know how to integrate 1 and ψ.

$$\int \psi d\psi = 1, \quad \int 1 d\psi = 0. \tag{3.15}$$

The integral is postulated to be *translationally invariant* under a shift by another Grassmann number η:

$$\int F(\psi + \eta) d\psi = \int F(\psi) d\psi. \tag{3.16}$$

This agrees with the expansion (3.14) if we set

$$\int \eta d\psi = 0. \tag{3.17}$$

In general for a collection of Grassmann numbers $(\psi_1, \ldots, \psi_N)$ we postulate that

$$\int \psi_i d\psi_j = \delta_{ij}. \tag{3.18}$$

There are no limits on these integrals and the integration is assumed to be a linear operation. The differential $d\psi$ is also a Grassmann number and so will anticommute with another Grassmann number ψ, hence $\int d\psi\psi = -1$. Remember here it is best to think of these integrals simply as operators on the Grassmann numbers. The integrals for $\overline{\psi}$ or any other Grassmann variable are identical. These integrals are simply assigned these values. A result we will use often is

$$\int \overline{\psi}\psi d\psi d\overline{\psi} = 1. \tag{3.19}$$

If the differentials or variables come in any other order there can be a change of sign. For example we will also invoke the result

$$\int \overline{\psi}\psi d\overline{\psi}d\psi = -1. \tag{3.20}$$

We need two more results before we can write down the path integral. The first is the resolution of the identity

$$I = \int |\psi\rangle\langle\overline{\psi}| e^{-\overline{\psi}\psi} d\overline{\psi}d\psi. \tag{3.21}$$

This can be seen using (3.9), (3.13) and (3.19) to give

$$\begin{aligned}
\int |\psi\rangle\langle\overline{\psi}| e^{-\overline{\psi}\psi} d\overline{\psi}d\psi &= \int |\psi\rangle\langle\overline{\psi}|(1-\overline{\psi}\psi)d\overline{\psi}d\psi \\
&= \int (|0\rangle - \psi\,|1\rangle)(\langle 0| - \langle 1|\,\overline{\psi})(1-\overline{\psi}\psi)d\overline{\psi}d\psi \\
&= \int \left(|0\rangle\langle 0| - |0\rangle\langle 1|\overline{\psi} - \psi|1\rangle\langle 0| + \psi|1\rangle\langle 1|\overline{\psi}\right)(1-\overline{\psi}\psi)d\overline{\psi}d\psi \\
&= \int \left(|0\rangle\langle 0| + \psi|1\rangle\langle 1|\overline{\psi}\right)(1-\overline{\psi}\psi)d\overline{\psi}d\psi \quad (\dagger) \\
&= -|0\rangle\langle 0| \int \overline{\psi}\psi d\overline{\psi}d\psi + |1\rangle\langle 1| \int \psi\overline{\psi}d\overline{\psi}d\psi \quad (\dagger\dagger) \\
&= |0\rangle\langle 0| + |1\rangle\langle 1| \\
&= I.
\end{aligned} \tag{3.22}$$

In step (†) and (††) recall that only $\overline{\psi}\psi = -\psi\overline{\psi}$ will have a non-zero integral and that Grassmann numbers square to zero. Finally, we will need that for any bosonic operator (an operator made of an even number of Fermi operators) Ω, the trace is given by[2]

$$\mathrm{Tr}\,\Omega = \int \langle -\overline{\psi}|\,\Omega\,|\psi\rangle\, e^{-\overline{\psi}\psi} d\overline{\psi}d\psi. \tag{3.23}$$

4 Fermionic Path Integral

We are now ready to map the quantum problem of Fermions to a path integral. Let us begin with

$$Z = \mathrm{Tr}\, e^{-\beta H}, \tag{4.1}$$

where H is a normal-ordered operator $H(\Psi^\dagger, \Psi)$. We will write the exponential as follows:

$$\begin{aligned}
e^{-\beta H} &= \lim_{N\to\infty} \left(e^{-\frac{\beta}{N}H}\right)^N \\
&= \lim_{N\to\infty} \underbrace{(1-\varepsilon H)\cdots(1-\varepsilon H)}_{N \text{ times}},
\end{aligned} \tag{4.2}$$

where we set $\varepsilon = \beta/N$. Now, using the fact that the the trace of the Boltzmann weight $e^{-\beta H}$ is the partition function (4.1), we will insert the identity between each N factor $N-1$ times using (3.21) and take the trace using (3.23) over $\overline{\psi}_0\psi_0$, giving us

$$\begin{aligned}
Z &= \mathrm{Tr}\, e^{-\beta H}, \\
&\approx \mathrm{Tr}\left(\underbrace{(1-\varepsilon H)\cdots(1-\varepsilon H)}_{N \text{ times}}\right) \\
&= \int \langle -\overline{\psi}_0|\,(1-\varepsilon H)I(1-\varepsilon H)I\cdots I(1-\varepsilon H)\,|\psi\rangle\, e^{-\overline{\psi}_0\psi_0} d\overline{\psi}_0 d\psi_0 \\
&= \int \langle -\overline{\psi}_0|\,(1-\varepsilon H)\,|\psi_{N-1}\rangle\, e^{-\overline{\psi}_{N-1}\psi_{N-1}} \langle\overline{\psi}_{N-1}|\,(1-\varepsilon H)\,|\psi_{N-2}\rangle\, e^{-\overline{\psi}_{N-2}\psi_{N-2}} \\
&\quad \times \langle\psi_{N-2}|\cdots|\psi_1\rangle\,\langle\overline{\psi}_1|\,(1-\varepsilon H)\,|\psi_0\rangle\, e^{-\overline{\psi}_0\psi_0} \prod_{i=0}^{N-1} d\overline{\psi}_i d\psi_i
\end{aligned} \tag{4.3}$$

[2] checked below formula but probably not worth including very similar to identity operator calculation

where are yet to take the limit $N \to \infty$. Note that $\varepsilon = \beta/N$ really has units of *time*$\hbar$, where we will set $\hbar = 1$. Now consider a single inner product in the above calculation, we can make the replacement

$$
\begin{aligned}
\langle \overline{\psi}_{i+1} | \, 1 - \varepsilon H(\Psi^\dagger, \Psi) \, | \psi_i \rangle &= \langle \overline{\psi}_{i+1} | \psi_i \rangle - \varepsilon \, \langle \overline{\psi}_{i+1} | \, H(\Psi^\dagger, \Psi) \, | \psi_i \rangle \\
&= e^{\overline{\psi}_{i+1}\psi_i} - \varepsilon \, \langle \overline{\psi}_{i+1} | \, H(\overline{\psi}_{i+1}, \psi_i) \, | \psi_i \rangle \\
&= e^{\overline{\psi}_{i+1}\psi_i} - \varepsilon H(\overline{\psi}_{i+1}, \psi_i) e^{\overline{\psi}_{i+1}\psi_i} \\
&= e^{\overline{\psi}_{i+1}\psi_i} (1 - \varepsilon H(\overline{\psi}_{i+1}, \psi_i)) \\
&= e^{\overline{\psi}_{i+1}\psi_i} e^{-\varepsilon H(\overline{\psi}_{i+1}, \psi_i)}
\end{aligned}
\tag{4.4}
$$

where in the last step we are anticipating the limit $\varepsilon \to 0$ and only keep terms linear or constant in ε. Now let us *define* additional pair of variables not to be integrated over by

$$
\begin{aligned}
\overline{\psi}_N &= -\overline{\psi}_0 \\
\psi_N &= -\psi_0.
\end{aligned}
\tag{4.5}
$$

The first of these equations allows us to replace the leftmost bra in (4.3) $\langle -\overline{\psi}_0 |$ with $\langle \overline{\psi}_N |$. The reason for doing this will become clear later. Putting together all the factors we end up with

$$
\begin{aligned}
Z &= \int \prod_{i=0}^{N-1} e^{\overline{\psi}_{i+1}\psi_i} e^{-\varepsilon H(\overline{\psi}_{i+1}, \psi_i)} e^{-\overline{\psi}_i \psi_i} d\overline{\psi}_i d\psi_i \\
&= \int \prod_{i=0}^{N-1} \exp\left[\left(\frac{\overline{\psi}_{i+1} - \overline{\psi}_i}{\varepsilon} \psi_i - H(\overline{\psi}_{i+1}, \psi_i) \right) \varepsilon \right] d\overline{\psi}_i d\psi_i \\
&= \int \exp\left[\sum_{i=0}^{N-1} \varepsilon \left(\frac{\overline{\psi}_{i+1} - \overline{\psi}_i}{\varepsilon} \psi_i - H(\overline{\psi}_{i+1}, \psi_i) \right) \right] \prod_{i=0}^{N-1} d\overline{\psi}_i d\psi_i
\end{aligned}
\tag{4.6}
$$

Now we are going to make perform a discrete version of integration by parts via

$$
\sum_{i=0}^{N-1} f_k (g_{k+1} - g_k) = (f_N g_N - f_0 g_0) - \sum_{i=0}^{N-1} g_{k+1} (f_{k+1} - f_k)
\tag{4.7}
$$

Therefore from (4.6) we have

$$
\begin{aligned}
Z &= \int \exp\left[\sum_{i=0}^{N-1} \left(\frac{\overline{\psi}_{i+1} - \overline{\psi}_i}{\varepsilon} \psi_i - H(\overline{\psi}_{i+1}, \psi_i) \right) \varepsilon \right] \prod_{i=0}^{N-1} d\overline{\psi}_i d\psi_i \\
&= \int \exp\left[\sum_{i=0}^{N-1} \left(\overline{\psi}_{i+1} \frac{\psi_{i+1} - \psi_i}{\varepsilon} - H(\overline{\psi}_{i+1}, \psi_i) \right) \varepsilon \right] \prod_{i=0}^{N-1} d\overline{\psi}_i d\psi_i
\end{aligned}
\tag{4.8}
$$

where we made use of (4.5) to eliminate the boundary terms. Now we need to take $N \to \infty$, which sends $\varepsilon = \beta/N \to 0$. So far the argument is mostly rigourous, the truely non-rigourous part comes in attempting to evaluate the limit. Let us think of the values $\psi_i, \overline{\psi}_i$ for $i = 0, \ldots, N-1$ as constituting values of a path $\psi(\tau)$ at the points $\tau_i = i\varepsilon = i\beta/N$, so

$$
\psi_i(\tau) = \psi(i\beta/N).
\tag{4.9}
$$

Since the distance between τ_i and τ_{i+1} is ε, the term $\frac{\psi_{i+1} - \psi_i}{\varepsilon}$ is an approximation to the derivative of $\psi(\tau)$ with respect to τ. Meanwhile the sum over i in the right handside of the exponent is an approximation of an integral. Thus is we then take the limit, in a totally non-rigourous fashion we obtain

$$
Z \simeq \int e^{S(\overline{\psi}, \psi)} [\mathcal{D}\overline{\psi} \mathcal{D}\psi],
\tag{4.10}
$$

where

$$
S = \int_0^\beta \left(\overline{\psi}(\tau) \left(-\frac{\partial}{\partial \tau} \right) \psi(\tau) \right) - H(\overline{\psi}(\tau), \psi(\tau)) d\tau.
\tag{4.11}
$$

The step in taking the limit $N \to \infty$ leading to the continuum form of the action (4.11) needs some explanation. With all the factors of ε present we do appear to get the continuum expression in the last formula. However, the notion of replacing differences by derivatives is purely symbolic for Grassmann variables. There is no sense in which $\overline{\psi}_{i+1} - \overline{\psi}_i$ is small, in fact the objects have no numerical values. What we really mean is that when evaluated in terms for ordinary numbers, the Grassman integral will give exact results for anything one wishes to calculate, such as the Free energy. With this approximation only quantities insensitive to high frequences (in Fourier space) will be given correctly. The free energy will come out wrong, but the correlation functions will be correctly reproduced (what we're interested in) because these are given as derivatives of the free energy and these derivatives make the integrals sufficiently insensitive to high frequencies. It is in this sense that we are replacing $H(\overline{\psi}_{i+1}, \psi_i) \to H(\overline{\psi}(\tau+\varepsilon), \psi(\tau))$ by $H(\overline{\psi}(\tau), \psi(\tau))$ in same spirit.

Now all we have to do is substitute the Hamiltonian for the 2D Ising model in terms of Majorana fermions (transformations of the Pauli operators) and we have our path integral for our theory to study the conformal field theory. Let's recall our usual Dirac fermion operators Ψ, and $\Psi^\dagger$. Recall these obey

$$\{\Psi, \Psi^\dagger\} = 1, \{\Psi, \Psi\} = \{\Psi^\dagger, \Psi^\dagger\} = 0 \tag{4.12}$$

We will consider the combinations

$$\begin{aligned} \psi_1 &= \frac{\Psi + \Psi^\dagger}{\sqrt{2}}, \quad \Psi = \frac{\psi_1 + i\psi_2}{\sqrt{2}} \\ \psi_2 &= \frac{\Psi - \Psi^\dagger}{\sqrt{2}i}, \quad \Psi^\dagger = \frac{\psi_1 - i\psi_2}{\sqrt{2}} \end{aligned} \tag{4.13}$$

which obey

$$\{\psi_i, \psi_j\} = \delta_{ij}. \tag{4.14}$$

In general N fermions ψ_i for $i = 1, \dots, n$ obeying the anticommutation relations (4.14) are called *Majorana Fermions*. Imagine that at each site we have a pair of Majorana fermions $\psi_1(n)$ and $\psi_2(n)$. These live in a two-dimensional space, as can bee seen by considering the Dirac fermions $\Psi(n), \Psi^\dagger(n)$ we can construct from them. The two states corresponding to $n_F = 0, 1$, on the full lattice the fermions will need a Hilbert space of dimension 2^N. We can define these Majorana fermions on the lattice via the *Jordan-Wigner Transformation*

$$\begin{aligned} \psi_1(n) &= \begin{cases} \frac{1}{\sqrt{2}} \left(\prod_{l=1}^{n-1} \sigma_1(l)\right) \sigma_2(n), & \text{if } n > 1, \\ \frac{1}{\sqrt{2}} \sigma_2(1), & \text{if } n = 1. \end{cases} \\ \psi_2(n) &= \begin{cases} \frac{1}{\sqrt{2}} \left(\prod_{l=1}^{n-1} \sigma_1(l)\right) \sigma_3(n), & \text{if } n > 1, \\ \frac{1}{\sqrt{2}} \sigma_3(1), & \text{if } n = 1. \end{cases} \end{aligned} \tag{4.15}$$

This implies that

$$\{\psi_i(n), \psi_j(m)\} = \delta_{ij}\delta_{nm}. \tag{4.16}$$

This transformation is to ensure that fermions at different sites will anticommute. While the transformation is good ensuring global anticommutation relations, simple operators involving a few spins will typically involve a large number of fermions. Amazingly though, something very nice happens for our two operators that appear in our Hamiltonian

$$\begin{aligned} \sigma_1(n) &= -2i\psi_1(n)\psi_2(n) \\ \sigma_3(n)\sigma_3(n+1) &= 2i\psi_1(n)\psi_2(n+1). \end{aligned} \tag{4.17}$$

Therefore, we can now rexpress our Hamiltonian in terms of Majorana fermions

$$\begin{aligned} H &= \sum_{n=1}^{N} -\lambda\sigma_1(n) - \sigma_3(n)\sigma_3(n+1) \\ &= \sum_{n=1}^{N} 2i\lambda\psi_1(n)\psi_2(n) - 2i\psi_1(n)\psi_2(n+1) \\ &= \sum_{n=1}^{N} 2i(\lambda - 1)\psi_1(n)\psi_2(n) - 2i\psi_1(n)\left[\psi_2(n+1) - \psi_2(n)\right] \end{aligned} \tag{4.18}$$

with the constraint $\psi(N+1) = \mp\psi(1)$. Now remember we have not taken the continuum limit of $a \to 0$ yet. If we want to consider the correlators in the continuum theory between points a distance x apart, we must chooes n such that $na = x$.

$$\langle \psi(0)\psi(n) \rangle = \frac{e^{-n/\xi(\lambda)}}{n} \tag{4.19}$$

where we note that $d = 2$ and $\nu = 1$ so $\xi(\lambda(a)) = 1/ma$ which we won't justify here, but see [KBS10]. We are going to define the *renormalized continuum fields* $\psi_r(x)$ as

$$\psi_r(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x}{a}\right). \tag{4.20}$$

To avoid a massive detour in this talk, we are simplying going to assert that $\lambda = 1 - ma$, this may be physically motivated based on particle physics, see [Sha17]. This process of rescaling the fields and varying the coupling λ and a to ensure the physical quantities are left fixed and finite in terms of the laboratory length x and mass m as we let $a \to 0$ is called taking the continuum limit. It follows that the renormalized fields obey the anticommutation rules in this limit

$$\{\psi_{\alpha r}(x), \psi_{\beta r}(x')\} = \delta_{\alpha\beta}(x - x'), \tag{4.21}$$

and our correlators have the form

$$\langle \psi_r(0)\psi_r(x) \rangle = \frac{e^{-mx}}{x}. \tag{4.22}$$

In the next talks though we will be interested in the correlation functions at the critical point $\lambda = 1$, which will look like power law functions.

Now we will take our Hamiltonian on the lattice in terms of the Majorana fermions (4.14) and build towards expressing these in the continuum limit

$$\begin{aligned} H &= \sum_n 2i(\lambda - 1)\psi_1(n)\psi_2(n) - 2i\psi_1(n)\left[\psi_2(n+1) - \psi_2(n)\right] \\ &= (\lambda - 1)\sum_n i\psi_1(n)\psi_2(n) + i\psi_1(n)\psi_2(n) \\ &\quad - i\sum_n \left(\psi_1(n)\psi_2(n+1) + \psi_1(n)\psi_2(n+1) - \psi_1(n)\psi_2(n) - \psi_1(n)\psi_2(n)\right) \end{aligned} \tag{4.23}$$

Now recall that $\lambda = 1 - ma$. We will also use the fact that $\psi_1(n)\psi_2(n) = -\psi_2(n)\psi_1(n)$ and we are going to add and subtract $i(\psi_1(n)\psi_2(n) - \psi_2(n)\psi_1(n))$ and to the Hamiltonian and bring them inside term without m. Doing all this gives us

$$\begin{aligned} H &= -a\sum_n m\left(i\psi_1(n)\psi_2(n) - i\psi_2(n)\psi_1(n)\right) \\ &\quad - i\sum_n \left(\psi_1(n)\psi_2(n+1) + \psi_1(n)\psi_2(n+1) - \psi_1(n)\psi_2(n) - \psi_1(n)\psi_2(n)\right) \\ &\quad + \sum_n i(\psi_1(n)\psi_2(n) - \psi_2(n)\psi_1(n)) - i(\psi_1(n)\psi_2(n) - \psi_2(n)\psi_1(n)) \\ &= -a\sum_n m\left(i\psi_1(n)\psi_2(n) - i\psi_2(n)\psi_1(n)\right) \\ &\quad - i\sum_n \psi_1(n)\psi_2(n+1) - \psi_1(n)\psi_2(n) - \psi_2(n+1)\psi_1(n) + \psi_2(n)\psi_1(n) \\ &= -a\sum_n m\left(i\psi_1(n)\psi_2(n) - i\psi_2(n)\psi_1(n)\right) \\ &\quad - i\sum_n \psi_1(n)(\psi_2(n+1) - \psi_2(n)) - (\psi_2(n+1) - \psi_2(n))\psi_1(n). \end{aligned} \tag{4.24}$$

Now we will again do a summation by parts on the term $(\psi_2(n+1) - \psi_2(n))\psi_1(n)$, noting that the boundary terms vanish. We will define $\Delta f(n) = f(n+1) - f(n)$ as the difference operator and then we obtain

$$\begin{aligned}
H &= -a \sum_n m \left(i\psi_1(n)\psi_2(n) - i\psi_2(n)\psi_1(n) \right) \\
&\quad - i \sum_n \psi_1(n)(\psi_2(n+1) - \psi_2(n)) - \psi_2(n+1)(\psi_1(n+1) - \psi_1(n)) \\
&= -a \sum_n m \left(i\psi_1(n)\psi_2(n) - i\psi_2(n)\psi_1(n) \right) \\
&\quad + a \sum_n \psi_1(n) \left(\frac{-i\Delta\psi_2(n)}{a} \right) + \psi_2(n+1) \left(\frac{-i\Delta\psi_1(n)}{a} \right)
\end{aligned} \tag{4.25}$$

Now we will take the continuum limit $a \to 0$. In doing so we are going to trade $\psi \to \psi_r$, $a\sum_n \to \int$, $\frac{\Delta\psi}{a} \to \frac{\partial\psi}{\partial x}$ since $x \gg a$. We will ignore the subscript r the renormalised fields, but we are left with

$$\begin{aligned}
H_r = \frac{H}{2a} &= \frac{1}{2} \int -mi\psi_1(x)\psi_2(x) + mi\psi_2(x)\psi_1(x) dx \\
&+ \frac{1}{2} \int \psi_1(x) \left(-i\frac{\partial}{\partial x} \right) \psi_2(x) + \psi_2(x) \left(-i\frac{\partial}{\partial x} \right) \psi_1(x) dx
\end{aligned} \tag{4.26}$$

Therefore looking back at our action (4.11) we have

$$S[\overline{\psi}, \psi] = \int \left(\overline{\psi}(x,\tau) \left(-\frac{\partial}{\partial \tau} \right) \psi(x,\tau) \right) - H(\overline{\psi}(x,\tau), \psi(x,\tau)) d\tau dx. \tag{4.27}$$

Clearly we have been a bit sloppy in remembering that these operators also depend on "time" τ, so really we should have

$$\begin{aligned}
H_r &= \frac{1}{2} \int -mi\psi_1(x,\tau)\psi_2(x,\tau) \ + mi\psi_2(x,\tau)\psi_1(x,\tau) \ dx \\
&+ \frac{1}{2} \int \psi_1(x,\tau) \left(-i\frac{\partial}{\partial x} \right) \psi_2(x,\tau) + \psi_2(x,\tau) \left(-i\frac{\partial}{\partial x} \right) \psi_1(x,\tau) dx
\end{aligned} \tag{4.28}$$

Furthermore note that we have an integral for an action over x and τ now. Recall that in the path integral derivation we are looking for stationary points of point particles on paths $\mathcal{P}$ in the action $S : \mathcal{P} \to \mathbb{R}$, so the action is an integral over time. In the case of quantum field theory, we are looking for a metric for a manifold $\mathcal{M}$ that is a stationary point of the action $S : \mathcal{M} \to \mathbb{R}$ so now we have an integral over spacetime, as in the ϕ^4 theory mentioned. Identifying $\overline{\psi} \to \psi_2$ and $\psi \to \psi_1$ we obtain

$$\begin{aligned}
S[\overline{\psi}, \psi] &= \int \left(\overline{\psi}(x,\tau) \left(-\frac{\partial}{\partial \tau} \right) \psi(x,\tau) \right) - H(\overline{\psi}(x,\tau), \psi(x,\tau)) d\tau dx \\
&= \int \left(\overline{\psi}(x,\tau) \left(-\frac{\partial}{\partial \tau} \right) \psi(x,\tau) \right) - \frac{1}{2} mi\psi(x,\tau)\overline{\psi}(x,\tau) \ + \frac{1}{2} mi\overline{\psi}(x,\tau)\psi(x,\tau) \\
&- \frac{1}{2}\psi(x,\tau) \left(-i\frac{\partial}{\partial x} \right) \overline{\psi}(x,\tau) - \frac{1}{2}\overline{\psi}(x,\tau) \left(-i\frac{\partial}{\partial x} \right) \psi(x,\tau) dx d\tau
\end{aligned} \tag{4.29}$$

Now we follow [Mus20] for transformation into complex coordinates

$$\begin{aligned}
\Psi(z,\bar{z}) &= \frac{\psi(x,\tau) + i\overline{\psi}(x,\tau)}{\sqrt{2}}, \\
\overline{\Psi}(z,\bar{z}) &= \frac{\psi(x,\tau) - i\overline{\psi}(x,\tau)}{\sqrt{2}} \\
\frac{\partial}{\partial \bar{z}} &= \frac{1}{2} \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial \tau} \right) \\
\frac{\partial}{\partial z} &= \frac{1}{2} \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial \tau} \right).
\end{aligned} \tag{4.30}$$

After a tediuous calculation one arrives as the free fermion action (up to a prefactor)

$$S = \int \Psi(z,\bar{z})\frac{\partial}{\partial \bar{z}}\Psi(z,\bar{z}) + \overline{\Psi}(z,\bar{z})\frac{\partial}{\partial z}\overline{\Psi}(z,\bar{z}) + im\overline{\Psi}(z,\bar{z})\Psi(z,\bar{z})dzd\bar{z}, \tag{4.31}$$

where recall the critical point is $m = 0$. So our action for the conformal field theory will be

$$S = \int \Psi(z,\bar{z})\frac{\partial}{\partial \bar{z}}\Psi(z,\bar{z}) + \overline{\Psi}(z,\bar{z})\frac{\partial}{\partial z}\overline{\Psi}(z,\bar{z})dzd\bar{z}, \tag{4.32}$$

which is where we will pick up as the starting point for conformal field theory next time.

References

[Hal13] Brian C. Hall. *Quantum Theory for Mathematicans.* Springer, 2013.

[KBS10] Peter Kopietz, Lorenz Bartosh, and Florian Schütz. *Introduction to the Functional Renormalization Group.* Springer, 2010.

[Mus20] G. Mussardo. *Statistical Field Theory: An Introduction to Exactly Solved Models in Statistical Physics.* Oxford University Press, 2020.

[Sha17] R. Shankar. *Quantum Field Theory and Condensed Matter.* Cambridge University Press, 2017.

[Vol21] Ray Volkas. PHYC90007 Quantum Mechanics, 2021.