Statistical Field Theory 4

Conformal Transformations and the Virasoro Algebra

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21 March 2024

1 Introduction

So far in our story we have taken a 2D classical Ising model in terms of spins s_i, quantised it to obtain spin operators $\hat{\sigma}_i$ that give the spins s_i as Eigenvalues. We then took the lattice spacing a to zero $a \to 0$ and the lattice size N to infinity $N \to \infty$ and set the temperature T to be the critical temperature $T = T_c$, and transformed our operators $\hat{\sigma} \to \Psi$. These operators Ψ are examples of (Fermionic) *Quantum Fields* $\phi : \mathbb{R}^d \to \mathbb{R}^d$ now acting on a d-dimensional (in our case $d = 2$) space-time manifold. Before we can use conformal symmetry to solve correlation functions of these operators, we need to find out conformal symmetry is! Much of the content here can be found in [Ger21, Rid13] and [Sch08].

2 What are Conformal Transformations?

Classically, a field theory refers to a construction of the dynamics of a field, i.e., a specification of how a field changes with time or with respect to other independent physical variables on which the field depends. Fields are modelled as vector (or tensor) valued functions on space-time. Usually this is done by writing an action for the field, and treating it as a classical mechanical system with an infinite number of degrees of freedom. In a quantum field theory, these functions become operators on the quantum state space $\mathcal{S}$. Conformal field theories are quantum field theories which are invariant under angle-preserving transformations. These transformations preserve the angle between two non-zero vectors in the field theory. Due to this invariance, conformal field theories do not require a Lagrangian or Hamiltonian description, however our example of the Ising model we saw so far did fit this bill. Let's take a step back from Quantum Fields and think about field theory and conformal transformations classically for a minute.

Definition 2.1. A *semi-Riemannian manifold* is a pair (M, g) consisting of a smooth manifold M of dimension d and a smooth tensor field g called the *metric tensor* that measures the infinitesimal distance between points on M that is symmetric and non-degenerate.

We will adopt the standard physics convention of representing vectors and matrices by their components. The *metric tensor* $g_{\mu\nu}$ is a matrix of dimension d that measures the infinitesimal distance between space-time points in a field theory. For $d = 2$, The infinitesimal distance squared is therefore

$$ds^2 = g_{11}(dx^1)^2 + g_{12}dx^1dx^2 + g_{21}dx^2dx^1 + g_{22}(dx^2)^2 \tag{2.1}$$

If we choose the Euclidean metric $\delta_{\mu\nu}$ we obtain the expected Pythagorean theorem

$$ds^2 = (dx^1)^2 + (dx^2)^2$$

We will also make use of Einstein summation notation, so (2.1) will be expressed as

$$ds^2 = \sum_{\mu,\nu=1}^{2} g_{\mu\nu}dx^\mu dx^\nu \equiv g_{\mu\nu}dx^\mu dx^\nu.$$

The metric tensor also can take a vector from a vector space to its dual space: $x_\mu = g_{\mu\nu}x^\nu$. The dot product in this special notation is therefore

$$x \cdot x = x^\mu g_{\mu\nu} x^\nu \tag{2.2}$$

We also denote the derivative $\frac{\partial}{\partial x^\mu}$ as ∂_μ and the norm as $\|x\| = (x \cdot x)^{1/2}$.

Now we need to discuss the algebra of conformal transformations that will be used to constrain our correlation functions. We will first classically derive how the conformal transformations infinitesimally affect a function on $\mathbb{R}^d$, then take $d = 2$, and using $\mathbb{R}^2 \cong \mathbb{C}$, show that these transformations become holomorphic and antiholomorphic transformations on the complex plane. In formally a conformal transformation is one which preserves, angles.

Definition 2.2. A *conformal transformation* to be a transformation $x \mapsto x'$ which preserves the metric up to a strictly positive, non-zero scale factor

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x), \tag{2.3}$$

where the scale factor depends on x.

Clearly (2.3) preserves angles. Consider two vectors x^μ and y^ν in the field theory over the field $\mathbb{R}^d$ where d is the dimension. If they are both transformed conformally to a point z so that (2.3) applies, then the angle between these vectors will transform as

$$\begin{aligned}
\cos(\theta') &= \frac{x' \cdot y'}{\|x'\| \, \|y'\|} \\
&= \frac{g_{\mu\nu}(z') x'^\mu y'^\nu}{\left(g_{\mu\nu}(z') x'^\mu x'^\nu\right)^{1/2} \left(g_{\mu\nu}(z') y'^\mu y'^\nu\right)^{1/2}} \\
&= \frac{\Lambda(z) g_{\mu\nu}(z) x^\mu y^\nu}{\left(\Lambda(z) g_{\mu\nu}(z) x^\mu x^\nu\right)^{1/2} \left(\Lambda(z) g_{\mu\nu}(z) y^\mu y^\nu\right)^{1/2}} \\
&= \frac{x \cdot y}{\|x\| \, \|y\|} \\
&= \cos(\theta).
\end{aligned} \tag{2.4}$$

While the above (2.4) applies to a metric that depends on the coordinates x, we will assume the metric now to be constant $g_{\mu\nu}(x) = g_{\mu\nu}$, however we will not assume that the transformed metric $g'_{\mu\nu}(x)$ is, in general.

To derive the conformal transformations infinitesimally, we will expand our vector $x'^\mu = x^\mu + \varepsilon^\mu(x)$ and

Lemma 2.1. *If we let* $\Lambda(x) = 1 - \Omega(x)$ *and expand our vector* $x'^\mu = x^\mu + \varepsilon^\mu(x)$ *we obtain the* infinitesimal conformal equation

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \Omega g_{\mu\nu}. \tag{2.5}$$

Proof. Observe that the infinitesimal length squared of a vector $\mathrm{d}s^2 = g'_{\mu\nu}\mathrm{d}x'^\mu \mathrm{d}x'^\nu$ between two points does not depend on our choice of coordinates. Therefore, $\mathrm{d}s^2 = g'_{\mu\nu}\mathrm{d}x'^\mu \mathrm{d}x'^\nu = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$. This leads to

$$\begin{aligned}
g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = g'_{\mu\nu}\mathrm{d}x'^\mu \mathrm{d}x'^\nu &= g'_{\mu\nu}\left(\mathrm{d}x^\mu + \partial_\rho \varepsilon^\mu \mathrm{d}x^\rho\right)\left(\mathrm{d}x^\nu + \partial_\sigma \varepsilon^\nu \mathrm{d}x^\sigma\right) \\
&= g'_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \partial_\rho g'_{\mu\nu}\varepsilon^\mu \mathrm{d}x^\rho \mathrm{d}x^\nu + \partial_\sigma g'_{\mu\nu}\varepsilon^\nu \mathrm{d}x^\mu \mathrm{d}x^\sigma \\
&= g'_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \partial_\rho \varepsilon_\nu \mathrm{d}x^\rho \mathrm{d}x^\nu + \partial_\sigma \varepsilon_\mu \mathrm{d}x^\mu \mathrm{d}x^\sigma \\
&= \left(\Lambda(x) g_{\mu\nu} + \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu\right)\mathrm{d}x^\mu \mathrm{d}x^\nu,
\end{aligned} \tag{2.6}$$

where we note $g'_{\mu\nu} = g'_{\nu\mu}$ since $g'_{\mu\nu}$ is symmetric, and we can let $\rho = \mu$ and $\sigma = \nu$, as these indices are summed over. We also safely ignored the infinitesimal squared term. While we used the primed metric to lower an index on an unprimed quantity. The error in doing so is also infinitesimal squared, and can be safely ignored. For the equality to hold we must have

$$g_{\mu\nu} = \Lambda(x) g_{\mu\nu} + \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu. \tag{2.7}$$

Substituting $\Lambda(x) = 1 - \Omega(x)$ gives the desired result. □

Now if we multiply both sides of (2.5) by $g^{\nu\mu}$, the inverse matrix of $g_{\nu\mu}$, we obtain

$$\begin{aligned} g^{\nu\mu}\partial_\mu\varepsilon_\nu + g^{\nu\mu}\partial_\nu\varepsilon_\mu &= \Omega g^{\nu\mu}g_{\mu\nu} \\ \partial^\nu\varepsilon_\nu + \partial^\mu\varepsilon_\mu &= \Omega d \\ \frac{2}{d}(\partial\cdot\varepsilon) &= \Omega, \end{aligned} \tag{2.8}$$

where $d = \delta^\mu_\mu$ is the dimension. We can see already from (2.8) that $d = 2$ appears to be of special interest, so we will take the hint and set $d = 2$, primarily because this is the setting that connects to all our hard work so far.

3 The Two-Dimensional Conformal Algebra

We will now derive the conformal transformations in two dimensions to create an algebra, namely the *conformal algebra*. In two dimensions with the Euclidean metric, this algebra is two commuting copies of the *Witt algebra*.

For $d = 2$, the constraints (2.5) and (2.8) heavily constrain the infinitesimal conformal transformations, and combining them will lead to the two-dimensional conformal algebra. We will now solve these for the Euclidean metric $g_{\mu\nu} = \delta_{\mu\nu}$. Take the infinitesimal conformal equation (2.5) and substitute our calculation for the scale factor Ω as in (2.8). This gives us

$$\begin{aligned} \partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu &= \delta_{\mu\nu}\frac{2}{d}\left(\partial\cdot\varepsilon\right) \\ &= \delta_{\mu\nu}\delta^{\rho\sigma}\partial_\rho\varepsilon_\sigma. \end{aligned}$$

Expanding out our indices, we get

$$\begin{aligned} 2\partial_1\varepsilon_1 = \partial_1\varepsilon_2 + \partial_2\varepsilon_1 &= 2\partial_2\varepsilon_2 \quad \text{for } \mu = \nu, \\ \partial_1\varepsilon_2 &= -\partial_2\varepsilon_1 \quad \text{for } \mu \neq \nu. \end{aligned}$$

These are the Cauchy-Riemann equations for ε^1 and ε^2. We now make a change of variables, since $\mathbb{R}^2 \cong \mathbb{C}$ [Mur24]

$$\begin{aligned} \varepsilon = \varepsilon^1 + \mathrm{i}\varepsilon^2, &\qquad \bar\varepsilon = \varepsilon^1 - \mathrm{i}\varepsilon^2, \\ z = x^1 + \mathrm{i}x^2, &\qquad \bar z = x^1 - \mathrm{i}x^2, \\ \partial_1 = \partial + \overline{\partial}, &\qquad \partial_2 = \partial - \overline{\partial}. \end{aligned} \tag{3.1}$$

Where we have used the notation $\partial = \partial_z$ and $\overline{\partial} = \partial_{\bar z}$. In these coordinates the Cauchy-Riemann equations, now for ε and $\bar\varepsilon$, read $\partial\bar\varepsilon = 0$ and $\overline{\partial}\varepsilon = 0$. Therefore, we conclude that $\varepsilon = \varepsilon(z)$ is holomorphic and $\overline{\varepsilon} = \overline{\varepsilon}(\bar z)$ is antiholomorphic.

Using this, we can now define the generators of these conformal transformations. These will be differential operators, and they generate a Lie algebra called the *conformal algebra*. To construct these generators, we take a classical field $\phi(x'^\mu)$ (scalar function), which is now a function of two variables: z and $\bar z$. Under the change of coordinates, $x'^\mu = x^\mu + \varepsilon^\mu$, we perform an infinitesimal Taylor expansion to obtain

$$\phi(x'^\mu) = \phi(x^\mu + \varepsilon^\mu) = \phi(x^\mu) + \varepsilon^\mu\partial_\mu\phi(x^\mu) + \cdots. \tag{3.2}$$

The generators are given as $\varepsilon^\mu\partial_\mu$. From here we will denote the field $\phi(x^\mu)$ as $\phi(z,\bar z)$.

Definition 3.1. A lie algebra $\mathfrak{g}$ is a non-commutative, non-associative algebra equipped with a bilinear operation $[-,-] : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying for $a, b \in \mathbb{F}$ and $x, y, z \in \mathfrak{g}$

$$\begin{aligned} &1.\quad [ax+by, z] = a[x,z] + b[y,z] \text{ and } [x, ay+bz] = a[x,y] + b[x,z] \\ &2.\quad [x,y] = -[y,x] \\ &3.\quad [[x,y],z] + [[y,z],x] + [[z,x],y] = 0. \end{aligned} \tag{3.3}$$

As an exmaple the vector space of all $n \times n$ matrices with real entries becomes a Lie algebra when the bracket is given by the commutator for $A, B \in \mathfrak{gl}(n; \mathbb{R})$

$$[A, B] = AB - BA. \tag{3.4}$$

Lemma 3.1. *The operators $\ell_n = -z^{n+1}\partial$, $\bar{\ell}_n = -\bar{z}^{n+1}\bar{\partial}$ are the generators of infinitesimal conformal transformations, which are two copies of a Lie algebra known as the* Witt Algebra*, which satisfy the following commutation relations*

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n}, \quad [\ell_m, \bar{\ell}_n] = 0, \quad [\bar{\ell}_m, \bar{\ell}_n] = (m-n)\bar{\ell}_{m+n}. \tag{3.5}$$

Proof. Note that any infinitesimal holomorphic transformation may be expressed as

$$z' = z + \epsilon(z), \quad \varepsilon(z) = \sum_{n=-\infty}^{\infty} c_n z^{n+1} \tag{3.6}$$

where the infinitesimal variation $\varepsilon(z)$ is expanded in a Laurent series around $z = 0$. Suppose that we had an arbitrary dimensionless field $\phi(z', \bar{z}')$, where we adopt our coordinates as specified in (3.1). Making an infinitesimal transformation (3.6) in (3.2) gives

$$\phi(z', \bar{z}') = \phi(z, \bar{z}) - \varepsilon(z)\partial\phi(z, \bar{z}) - \bar{\varepsilon}(\bar{z})\bar{\partial}\phi(z, \bar{z}).$$

Setting $\delta\phi(z, \bar{z}) = \phi(z', \bar{z}') - \phi(z, \bar{z})$, we see

$$\begin{aligned} \delta\phi(z, \bar{z}) &= -\varepsilon(z)\partial\phi(z, \bar{z}) - \bar{\varepsilon}(\bar{z})\bar{\partial}\phi(z, \bar{z}) \\ &= \sum_{n=-\infty}^{\infty} \left(c_n \ell_n \phi(z, \bar{z}) + \bar{c}_n \bar{\ell}_n \phi(z, \bar{z})\right), \end{aligned} \tag{3.7}$$

where we identify the generators

$$\ell_n = -z^{n+1}\partial, \quad \bar{\ell}_n = -\bar{z}^{n+1}\bar{\partial}. \tag{3.8}$$

This tells us that ℓ_n and $\bar{\ell}_n$ are the generator of infinitesimal conformal transformations. Since these are differential operators, we can compute the commutators of these generators

$$\begin{aligned} [\ell_m, \ell_n] &= [-z^{m+1}\partial, -z^{n+1}\partial] \\ &= z^{m+1}\partial\left(z^{n+1}\partial\right) - z^{n+1}\partial\left(z^{m+1}\partial\right) \\ &= z^{m+1}\left((n+1)z^n\partial + z^{n+1}\partial^2\right) - z^{n+1}\left((m+1)z^m\partial + z^{m+1}\partial^2\right) \\ &= (n-m)z^{m+n+1}\partial \\ &= (m-n)\ell_{m+n}. \end{aligned} \tag{3.9}$$

The anti-holomorphic calculation is identical, and the reader can see from inspection that the holomorphic and anti-holomorphic generators commute, hence $[\ell_m, \bar{\ell}_n] = 0$. This gives us the conformal algebra in two dimensions as the direct sum of two infinite dimensional Lie algebras, each known as the *Witt Algebra.* □

In other words, the Witt algebra has the generators $(\ell_n)_{n\in\mathbb{Z}}$ as a complex vector space basis, satisfying (3.9). The algebra of infinitesimal conformal transformations with a Euclidean metric in two dimensions is infinite dimensional. Note that the Witt algebra (3.9) contains a finite dimensional subalgebra generated by $\{\ell_{-1}, \ell_0, \ell_1\}$.

There is a much longer story to be told here about this finite dimensional subalgebra, for our use however, we need only realise that the set of conformal transformations $\{\ell_{-1}, \bar{\ell}_{-1}, \ell_0, \bar{\ell}_0, \ell_1, \bar{\ell}_1\}$ encode the following:

- $-\left(\ell_{-1} + \bar{\ell}_{-1}\right)$, $-\mathrm{i}\left(\ell_{-1} + \bar{\ell}_{-1}\right)$ generate classical infinitesimal translations
- $-\left(\ell_0 + \bar{\ell}_0\right)$, $-\mathrm{i}\left(\ell_0 + \bar{\ell}_0\right)$ generate classical infinitesimal dilations and rotations

- $-\left(\ell_1+\bar{\ell}_1\right)$, $-\mathrm{i}\left(\ell_1+\bar{\ell}_1\right)$ generate classical infinitesimal special conformal transformations

Definition 3.2. A *global conformal transformations* is an injective holomorphic function which is defined on the entire plane $\mathbb{C}$ with at most on exceptional point

The exceptional point is taken to be $\{\infty\}$. These *global conformal transformations* can be exponentiated to give the *Möbius Transformations* on the Riemann sphere $\mathbb{C}\cup\{\infty\}$ which is our manifold M we will be using with the usual metric from now on.

Definition 3.3. The global conformal transformations defined on the Riemann sphere $\mathbb{C}\cup\{\infty\}$ are the Möbius transformations:

$$z \mapsto \frac{az+b}{cz+d} \quad \text{with} \quad a,b,c,d \in \mathbb{C}, \tag{3.10}$$

such that $ad-bc \neq 0$.

- Global translations are given by $c=0$ and $a,d=1$: $z\mapsto z+b$
- Global rotations and dilations are given by $b,c=0$ and $d=1$: $z\mapsto az$
- Global special conformal transformation (SCT) is given by $a=0$ and $b,d=1$: $z\mapsto \frac{1}{cz+1}$

The set of Möbius transformations is the set of all orientation preseving global conformal transformations. One last point to address before moving on is that the conformal group is often referred to as "infinite dimensional", but what does this mean? This is an artifact of physicists thinking and calculating infinitesimally (Witt algebra) but then talking and writing globally (Möbius transformations). In two-dimensional conformal field theory, usually only the infinitesimal conformal invariance of the system under consideration is used.

Now that we know the two-dimensional *conformal algebra*, which is two commuting copies of the *Witt algebra*, we need to quantise these symmetry generators so that they may act upon our quantum field theory.

4 The Virasoro Algebra

Thus far we have worked in a classical field theory, however in a quantum field theory, the symmetry generators become operators on the quantum state space $\mathcal{S}$. The goal here is now to take our algebra of conformal transformations in two dimensions, the Witt algebra (3.9), and derive the quantised version: the Virasoro algebra $\mathfrak{Vir}$.

In a conformally invariant quantum field theory, instead of the quantum state space admitting a representation of two copies of the Witt algebra, the projective state space will admit such a representation. As we wish to work in the state space $\mathcal{S}$, we will lift this projective representation to a representation of the central extension of the Witt algebra, giving the Virasoro algebra $\mathfrak{Vir}$.

Another reason one would ever want to do this is that the classical conformal symmetry no longer holds at the quantum level due to the presence of the trace anomaly. More precisely, the tracelessness of the quantum stress-energy tensor is incompatible with the normal ordering needed to define it. We haven't met the stress-energy tensor yet or normal ordering, so we will just treat this as a part of the canonical quantisation procedure.

Definition 4.1. For a general Lie algebra $\mathfrak{g}$, we define the *central extension* of $\mathfrak{g}$ to be $\mathfrak{g}'=\mathfrak{g}\oplus\mathbb{C}$ with commutation relations:

$$\begin{aligned} [x',y']_{\mathfrak{g}'} &= [x,y]_{\mathfrak{g}} + c\,p(x,y) \quad x',y'\in\mathfrak{g}',\ x,y\in\mathfrak{g}, \\ [x',c]_{\mathfrak{g}'} &= 0, \quad x'\in\mathfrak{g}',\ c\in\mathbb{C}, \end{aligned} \tag{4.1}$$

where $x'=x\oplus 0$, $c=0\oplus c$ in $\mathfrak{g}'$ for $x\in\mathfrak{g}, c\in\mathbb{C}$. Furthermore, $p:\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ is a bilinear function.

Definition 4.2. The Virasoro Algebra $\mathfrak{Vir}$ is a complex Lie algebra with infinite basis given by $\{L_n|n\in\mathbb{Z}\}\cup\{c\}$ with commutation relations

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{1}{12}(m^3-m)\delta_{m+n,0}c. \tag{4.2}$$

The constant c is called the *central charge* of our CFT.

Lemma 4.1. *The Virasoro Algebra* $\mathfrak{Vir}$ *is the cental extension of the Witt algebra.*

Proof. We denote the generators of the quantum conformal symmetry algebra by L_n. In terms of (4.1), $x', y' = L_m, L_n$ with $\mathfrak{g}' = \mathfrak{Vir}$ will be the Virasoro algebra modes, with $x, y = \ell_m, \ell_n$ as the Witt algebra modes. Using (4.1), the L_n modes will have the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + c\,p(m,n), \tag{4.3}$$

for some function p. Since the Lie bracket is antisymmetric, $p(m,n) = -p(n,m)$. We can show that by appropriately redefining the generators $L_m \mapsto \widehat{L}_m$, we can arrange for $p(n,0)$ and $p(1,-1)$ to be zero by setting

$$\widehat{L}_m = L_m + \frac{c\,p(n,0)}{n}, \text{ for } n \neq 0,$$
$$\widehat{L}_0 = L_0 + \frac{c\,p(1,-1)}{2}.$$

This is to provide the commutation relations

$$\begin{aligned} [\widehat{L}_m, \widehat{L}_0] &= mL_m + c\,p(m,0) = \widehat{L}_m \\ [\widehat{L}_1, \widehat{L}_{-1}] &= 2L_0 + c\,p(1,-1) = 2\widehat{L}_0. \end{aligned} \tag{4.4}$$

We will drop the hats, but the algebra from this point on in this thesis can be understood with this redefinition in mind. To determine $p(m,n)$, we will consider two Jacobi identities.

First consider the Jacobi identity:

$$0 = [[L_m, L_n], L_0] + [[L_n, L_0], L_m] + [[L_0, L_m], L_n] = (m+n)p(n,m).$$

When $m + n \neq 0$, we have $p(m,n) = 0$. Therefore, the only values to fix are $p(n,-n)$ for $n \geq 2$, since we set $p(1,-1) = 0$, and p is antisymmetric.

Now consider a second Jacobi identity:

$$\begin{aligned} [[L_{-n+1}, L_n], L_{-1}] + [[L_n, L_{-1}], L_{-n+1}] + [[L_{-1}, L_{-n+1}], L_n] &= 0 \\ c\,p(n-1,-n+1) + (n-2)c\,p(-n,n) &= 0. \end{aligned} \tag{4.5}$$

where we have made use of our redefinition (4.4). Rearranging (4.5), we obtain a recursion relation for $p(n,-n)$:

$$\begin{aligned} p(n,-n) &= \frac{n+1}{n-2}p(n-1,-n+1) \\ &= \frac{(n+1)n(n-1)\cdots 5\cdot 4}{(n-2)(n-3)(n-4)\cdots 2\cdot 1}p(2,-2) \\ &= \frac{(n+1)n(n-1)\cdots 5\cdot 1}{6(n-2)!}p(2,-2) \\ &= \frac{1}{6}(n+1)n(n-1)p(2,-2). \end{aligned}$$

Normalising $p(2,-2) = -\frac{1}{2}$, we have obtained

$$p(m,n) = \frac{1}{12}(m+1)m(m-1)\delta_{m+n,0},$$

which completes the calculation.

We have now arrived at the *Virasoro algebra* $\mathfrak{Vir}$ as the central extension of the Witt algebra. This algebra has basis $\{L_n\} \cup \{c\}$ for $n \in \mathbb{Z}$ and $c \in \mathbb{C}$ with commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m+1)m(m-1)\delta_{m+n,0}, \quad [L_m, c] = 0. \tag{4.6}$$

This calculation informs us that we have two commuting copies of the Virasoro algebra in our quantum conformal algebra. $\square$

Let us summarise what we have achieved so far. We started by defining a conformal transformation to be one that leaves the metric invariant up to a scale (2.3). This allowed us to derive the equation (2.5), and specialising to two dimensions, we found that the Witt algebra is an infinite dimensional Lie algebra that encodes infinitesimal conformal transformations. We finished by extending this algebra to a quantum field theory, where our algebra for infinitesimal conformal transformations is now the Virasoro algebra (4.6).

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