The Structure Constants of the Minimal Models

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Conformal Field Theory

- Quantum field theories with conformal symmetry imposed

- Area from Belavin, Polyakov and Zamalodchikov in 1984 where they applied representations of the Virasoro algebra to statistical models

- Goal of measuring correlation functions: how quantum fields interact with each other

- Conformal Symmetry assists in computing these correlation functions and in the case of 3 fields interacting conformal symmetry fixes correlators up to an unknown constant

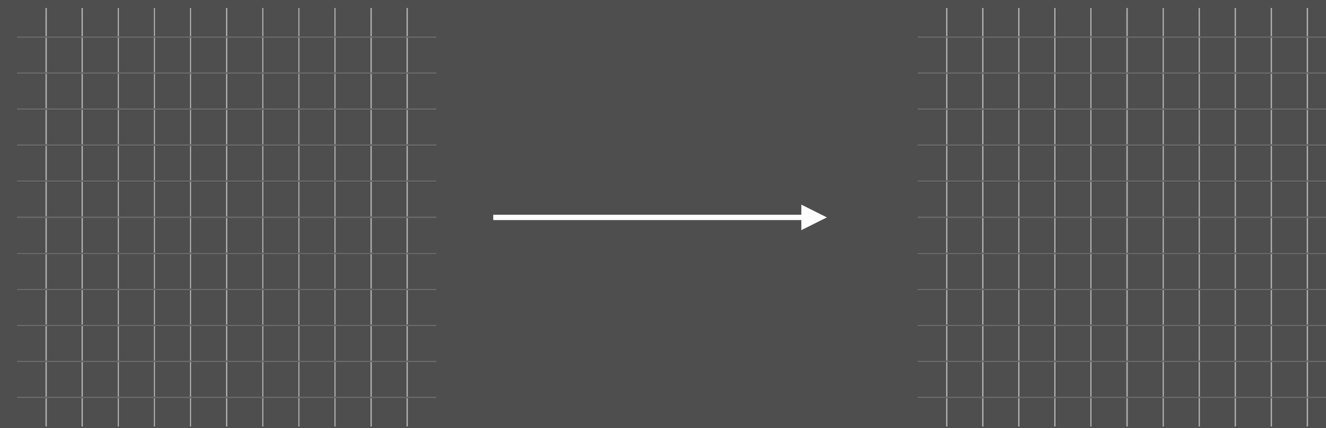
- We outline how to compute these constants for a class of CFTs called minimal models

- 2D CFTs on the Riemann Sphere $\mathbb{C} \cup \{\infty\}$ with independent coordinates $z, \bar{z}$

The Conformal Transformations in the complex plane

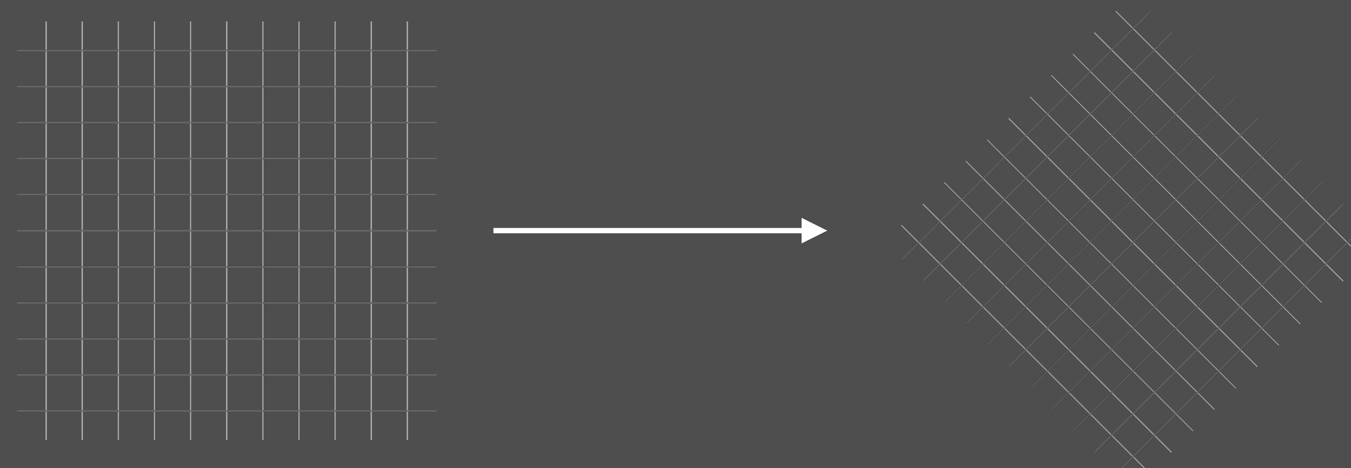
$(a, b, c \in \mathbb{C}, \theta \in \mathbb{R})$

- Translations



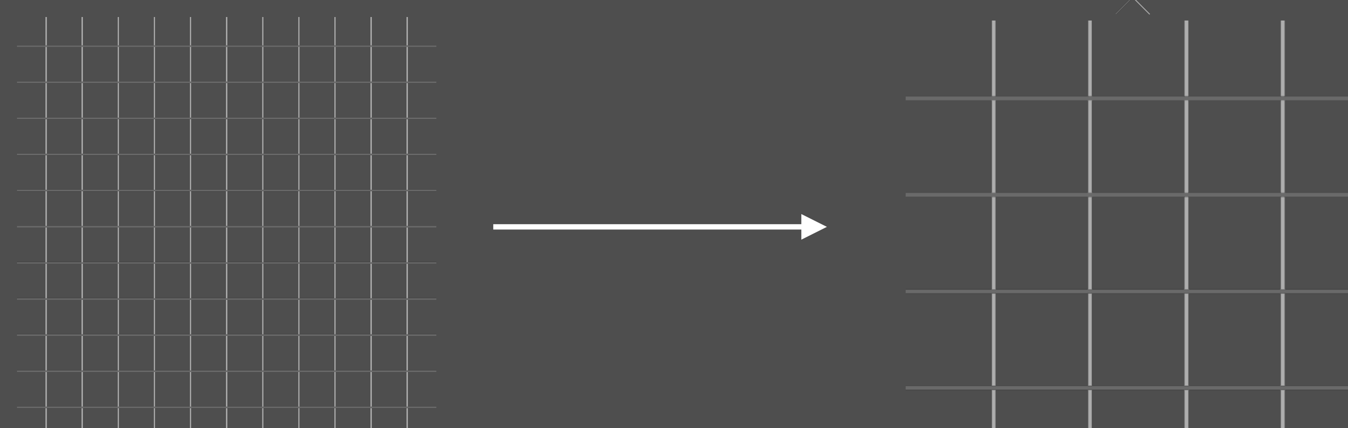
$z \mapsto z + b$

- Rotations



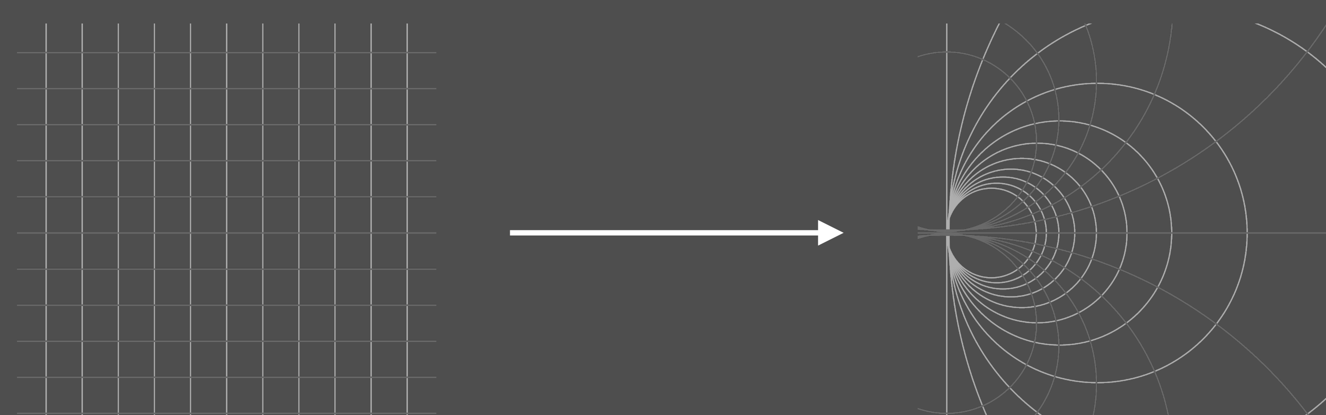
$z \mapsto e^{i\theta} z$

- Dilations



$z \mapsto |a| z$

- Special Conformal Transformations



$$z \mapsto \frac{1}{z^{-1} + c}$$

The Conformal Algebra

- The generators of these transformations form a Lie algebra called the Virasoro algebra $\mathfrak{Vir}$ with basis $\{L_n\}_{n\in\mathbb{Z}} \cup \{c\}$ and commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m+1)m(m-1)\delta_{m+n,0}, \quad [L_m, c] = 0$$

- $c \in \mathbb{C}$ is the central charge

- L_{-1}, L_0 and L_1 encode the global conformal transformations seen previously, together, they generate the Möbius transformations

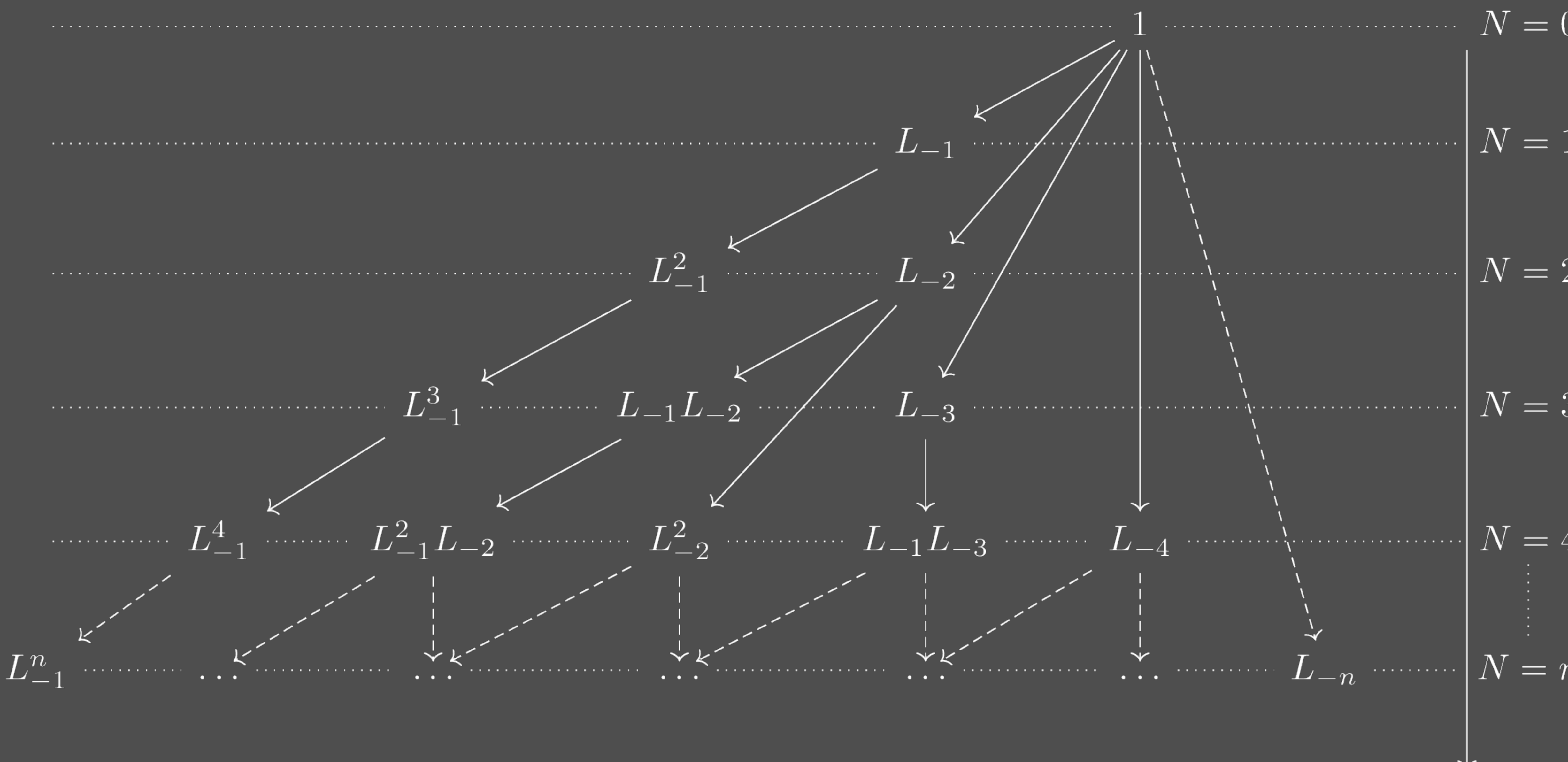
$$z \mapsto \frac{az+b}{cz+d}, \qquad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0$$

Minimal Models

- Virasoro minimal models $\mathrm{M}(p, q)$ with $p, q \geq 2$ co-prime and $p < q$
- Conformal fields $\phi_{r,s}(z, \bar{z})$ carry conformal dimensions $h_{r,s}$ and $\bar{h}_{r,s}$
- We consider holomorphic conformal fields as $\phi_{r,s}(z)$ with $h_{r,s}$, and tensor with antiholomorphic ones later
- Primary fields have a State field correspondence $\lim_{z\to 0} \phi_{r,s}(z)\,|\,0\rangle = |\,h_{r,s}\rangle$
- We create descendent fields from primary fields by acting with creation operators L_{-n} for $n > 0$, L_n for $n > 0$ are annihilation operators
- The energy-momentum field $T(z)$ is a conformal field that has conformal dimension 2 and Fourier decomposition with modes from the Virasoro algebra

$$\lim_{z\to 0} T(z)\,|\,0\rangle = L_{-2}\,|\,0\rangle$$

- Quantum state space constructed from highest weight irreducible representations of the Virasoro algebra $\mathfrak{Vir}$ generated from $|h_{r,s}\rangle$
- Basis for descendent states from a highest weight state



- Central charge is

$$c = 1 - 6\frac{(p-q)^2}{pq}$$

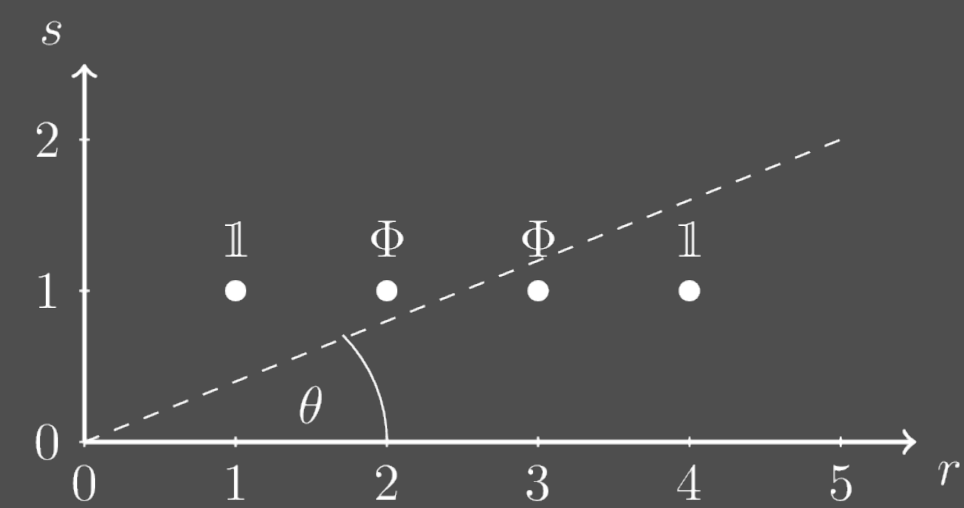
- Conformal dimension of $\phi_{r,s}$ is

$$h_{r,s} = \frac{(pr-qs)^2 - (p-q)^2}{4pq}$$

$$1 \le r < q, \quad 1 \le s < p$$

- Conformal fields are symmetric around dashed line via $\phi_{r,s} = \phi_{q-r,p-s}$

- One model introduced by BPZ $\mathrm{M}(2,5)$ has only two fields $\phi_{1,1}$ and $\phi_{2,1}$ and is identified with the Yang-Lee singularity in statistical mechanics



Operator Product Expansion

- We can expand products of fields as an operator product expansion, similar to a Laurent series of holomorphic functions

- A operator product expansion of primary fields has the form

$$\phi_1(z_1)\phi_2(z_2) = \sum_k \frac{C_{12}{}^{k}\phi_k(z_2)}{(z_1 - z_2)^{h_1+h_2-h_k}} + \ldots$$

- Operator product expansion of a primary conformal field $\phi(w)$ of conformal dimension h and the energy momentum field $T(z)$ is

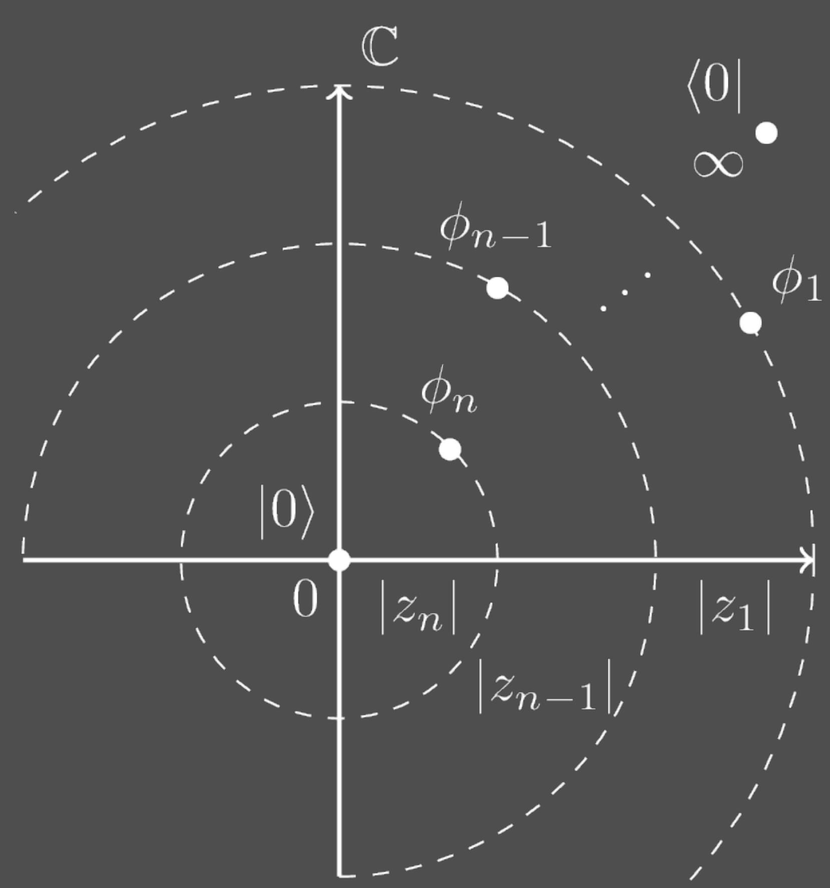
$$T(z)\phi(w) = \frac{h\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{(z-w)} + \cdots$$

Correlation Functions

- As mentioned, one of the goals of conformal field theory is to compute correlation functions: scalar product of conformal fields $\phi_k(z_k)$

- In 2D CFT time is measured radially from the origin

- Start with vacuum state $|0\rangle$, act with $\phi_n(z_n, \bar{z}_n)$ at time $|z_n|$, then with $\phi_{n-1}(z_{n-1}, \bar{z}_{n-1})$ at time $|z_{n-1}|$, finally with $\phi_1(z_1, \bar{z}_1)$ at time $|z_1|$, then evaluate the result with the linear functional $\langle 0|$ giving



$$\langle 0|\phi_1(z_1, \bar{z}_1)\ldots\phi_n(z_n, \bar{z}_n)|0\rangle, \quad |z_n| < \ldots < |z_1|$$

- Correlation functions are single valued, invariant under $z \mapsto e^{2\pi i}z$, $\bar{z} \mapsto e^{-2\pi i}\bar{z}$

- Physically measurable observables

Correlation Functions from Ward Identities

- Inserting a Virasoro mode L_m into a chiral correlation function gives a set of partial differential equations for $m = -1,0,1$ called Ward identities, the solutions include

$$\langle 0 | \phi_0(z_0) | 0 \rangle = \delta_{\phi_0 = \mathrm{id}}, \quad \langle 0 | \phi_0(z_0)\phi_1(z_1) | 0 \rangle = \frac{C_{01}\delta_{h_0=h_1}}{(z_0 - z_1)^{h_0+h_1}}$$

- The 3-point function has the form

$$\langle 0 | \phi_0(z_0)\phi_1(z_1)\phi_2(z_2) | 0 \rangle = \frac{C_{012}}{z_{01}^{h_0+h_1-h_2} z_{02}^{h_0-h_1+h_2} z_{12}^{-h_0+h_1+h_2}}$$

- The 4-point function has the form

$$\langle 0 | \phi_0(z_0)\phi_1(z_1)\phi_2(z_2)\phi_3(z_3) | 0 \rangle = F(z) \prod_{0 \le i < j \le 3} z_{ij}^{\mu_{ij}}, \quad \mu_{ij} = \frac{1}{3}\sum_{k=0}^{3} h_k - h_i - h_j$$

- $F(z)$ is an arbitrary function of the cross ratio

$$z = \frac{(z_0 - z_1)(z_2 - z_3)}{(z_0 - z_3)(z_1 - z_2)}$$

Benefit of Global Conformal Invariance

- We can send $z_1 \to 0,\ z_2 \to 1,\ z_3 \to \infty$ to make the cross ratio one variable

$$z = \frac{(z_0 - z_1)(z_2 - z_3)}{(z_0 - z_3)(z_1 - z_2)} \implies z = -z_0 \qquad z \mapsto \frac{az+b}{cz+d}$$

- We only need to consider 4 point functions of 1 variable

First Method: Ward Identities + OPEs

- The operator product expansion of the conformal fields $\phi_{2,1}(z)$, $\phi_{r,s}(w)$ is

$$\phi_{2,1}(z)\phi_{r,s}(w) = \frac{C_{(2,1)(r,s)}{}^{(r-1,s)}\phi_{r-1,s}(w)}{(z-w)^{h_{2,1}+h_{r,s}-h_{r-1,s}}} + \cdots + \frac{C_{(2,1)(r,s)}{}^{(r+1,s)}\phi_{r+1,s}(w)}{(z-w)^{h_{2,1}+h_{r,s}-h_{r+1,s}}} + \cdots$$

- We can fuse fields together using OPEs to reduce a 4-point function to 3-point functions, and substations of solutions of the Ward identities

$$\langle 0|\phi_{r_1,s_1}(\infty)\phi_{r_2,s_2}(1)\underbrace{\phi_{2,1}(z)\phi_{r_3,s_3}}_{\text{OPE}}(0)|0\rangle = \frac{C_{(2,1)(r_3,s_3)}{}^{(r_3-1,s_3)}}{z^{h_{2,1}+h_{r_3,s_3}-h_{r_3-1,s}}}\langle 0|\phi_{r_1,s_1}(\infty)\phi_{r_2,s_2}(1)\phi_{r_3-1,s_3}(0)|0\rangle + \cdots$$

$$+\frac{C_{(2,1)(r_3,s_3)}{}^{(r_3+1,s_3)}}{z^{h_{2,1}+h_{r_3,s_3}-h_{r_3+1,s}}}\langle 0|\phi_{r_1,s_1}(\infty)\phi_{r_2,s_2}(1)\phi_{r_3+1,s_3}(0)|0\rangle + \cdots$$

$$F^{(4)}(z,\bar{z}) = C_{(2,1)(r_3,s_3)}{}^{(r_3-1,s_3)}C_{(r_1,s_1)(r_2,s_2)(r_3-1,s_3)}|z|^{\alpha} + \cdots$$

$$+C_{(2,1)(r_3,s_3)}{}^{(r_3+1,s_3)}C_{(r_1,s_1)(r_2,s_2)(r_3+1,s_3)}|z|^{\beta} + \cdots$$

Second Method: Ward identities + Singular Vectors

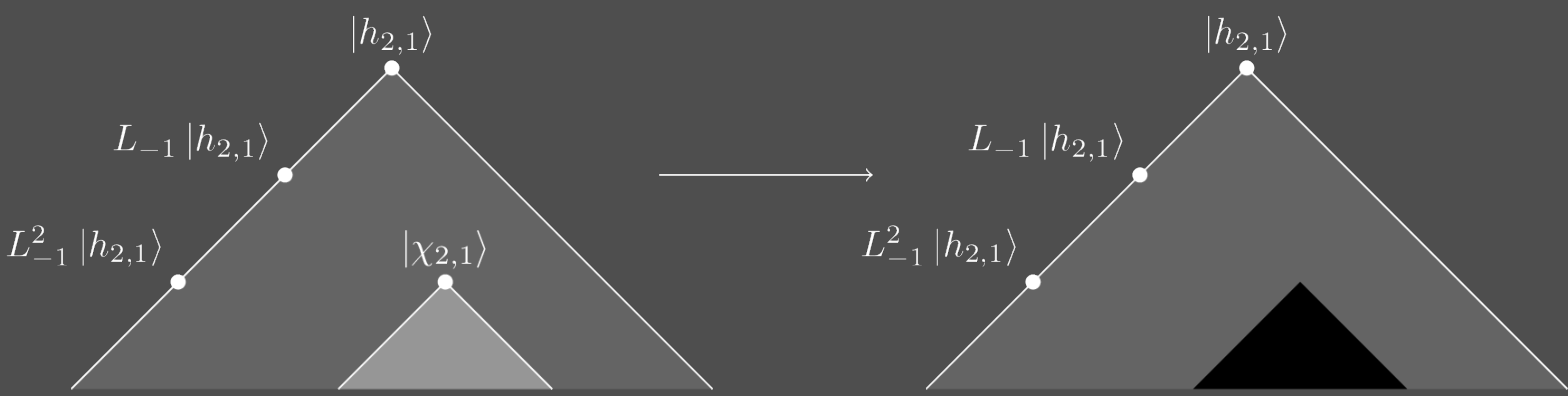
- A highest weight state $|h_{r,s}\rangle$ may have a descendent highest weight state $|\chi_{r,s}\rangle$ called a singular vector. For example

$$|\chi_{2,1}\rangle = \left(L_{-2} - \frac{q}{p}L_{-1}^2\right)|h_{2,1}\rangle$$

- Under the state-field correspondence this gives a singular field

$$\chi_{2,1}(z) = (L_{-2}\phi_{2,1})(z) - \frac{q}{p}\partial_z^2\phi_{2,1}(z)$$

- Both $|\chi_{2,1}\rangle$ and $\chi_{2,1}(z)$ are set to zero



- Inserting a singular vector into a correlation function, gives a differential equation

$$\langle 0 | \phi_0(z_0)\phi_1(z_1)\chi_{2,1}(z_2)\phi_3(z_3) | 0 \rangle = 0$$

$$\left(\frac{\partial^2}{\partial {z_2}^2} - \frac{p}{q} \sum_{j \neq 2} \left[\frac{1}{z_2 - z_j} \frac{\partial}{\partial z_j} + \frac{h_j}{(z_2 - z_j)^2} \right] \right) \langle 0 | \phi_0(z_0)\phi_1(z_1)\phi_{2,1}(z_2)\phi_3(z_3) | 0 \rangle = 0$$

- Ward identities give the form of the 4-point correlation function. Sending $z_1 \to 0,\, z_2 \to 1,\, z_3 \to \infty$ we obtain the BPZ equation

$$\left[\frac{q}{p} \partial_z^2 + \frac{2z - 1}{z(z-1)} \partial_z - \frac{h_3}{z^2} - \frac{h_1}{(z-1)^2} + \frac{h_2 + h_3 + h_1 - h_0}{z(z-1)} \right] H(z) = 0$$

- Where $H(z) = z^{\mu_{13}}(1 - z)^{\mu_{12}} F(z)$ for the arbitrary function of the cross ratio

- This equation can be transformed into a hypergeometric equation, with hypergeometric functions as solutions $F(A, B; C; z)$

- The two linearly independent solutions for the undetermined function of the cross ratio is

$$F_1(z) = z^{\frac{1}{2}\alpha}(1-z)^{\frac{1}{2}\gamma} F(A, B; C; z), \quad F_2(z) = z^{\frac{1}{2}\beta}(1-z)^{\frac{1}{2}\delta} F(A', B'; C'; z)$$

$$F^{(4)}(z, \bar{z}) = C_{1,1} \, |F_1(z)|^2 + C_{2,2} \, |F_2(z)|^2$$

- We now have two expansions for the correlation function around $z = 0$

$$F^{(4)}(z, \bar{z}) = C_{1,1} \, |z|^{\alpha} + C_{2,2} \, |z|^{\beta} + \dots$$

$$F^{(4)}(z, \bar{z}) = C_{(2,1)(r_3,s_3)}^{\quad (r_3+1,s_3)} C_{(r_1,s_1)(r_2,s_2)(r_3+1,s_3)} \, |z|^{\alpha} + \cdots$$

$$+ C_{(2,1)(r_3,s_3)}^{\quad (r_3-1,s_3)} C_{(r_1,s_1)(r_2,s_2)(r_3-1,s_3)} \, |z|^{\beta} + \cdots$$

- All parameters in expansions are functions of p, q, r, s

- Comparing the two expansions for our correlation function gives

$$F^{(4)}(z,\bar{z}) = C_{(2,1)(r_3,s_3)}^{\quad\;(r_3+1,s_3)} C_{(r_1,s_1)(r_2,s_2)(r_3+1,s_3)} \left| z \right|^{\alpha} \left| 1-z \right|^{\gamma} \left| F(A,B;C;z) \right|^2$$

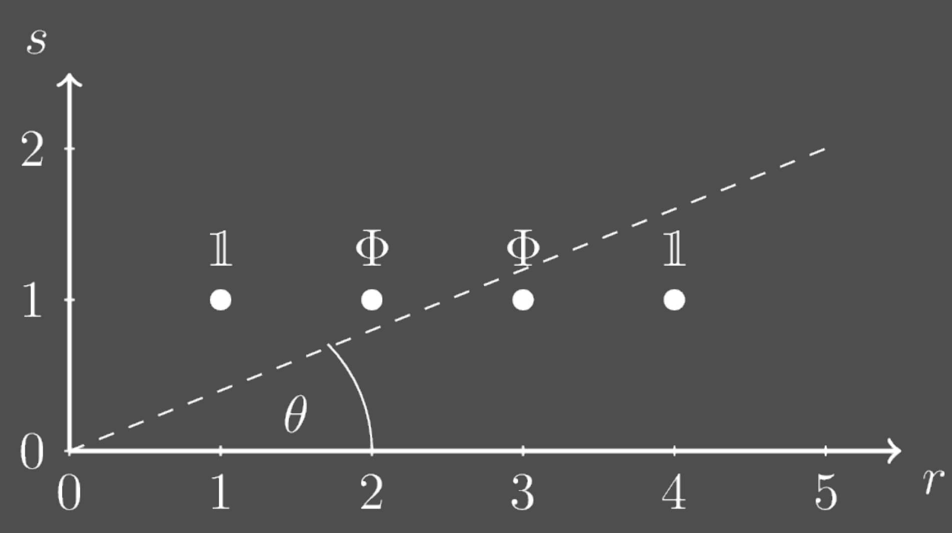
$$+ C_{(2,1)(r_3,s_3)}^{\quad\;(r_3-1,s_3)} C_{(r_1,s_1)(r_2,s_2)(r_3-1,s_3)} \left| z \right|^{\beta} \left| 1-z \right|^{\delta} \left| F(A',B';C';z) \right|^2$$

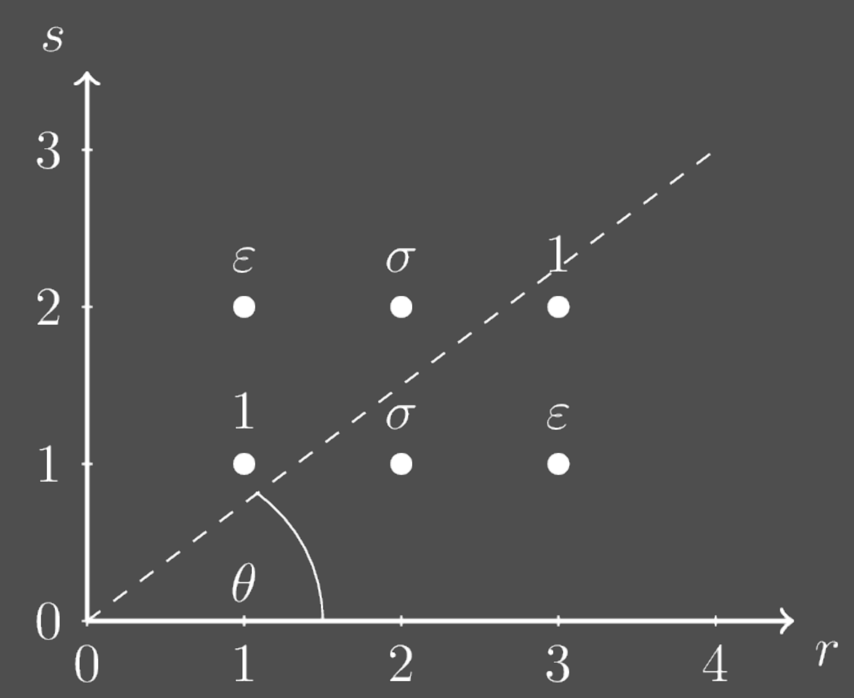
- Since this should be valid around $z = 1$, we can expand using hypergeometric identities, setting the non-single valued terms to zero giving the equation

$$\frac{C_{(2,1)(r_3,s_3)}^{\quad\;(r_3+1,s_3)} C_{(r_1,s_1)(r_2,s_2)(r_3+1,s_3)}}{C_{(2,1)(r_3,s_3)}^{\quad\;(r_3-1,s_3)} C_{(r_1,s_1)(r_2,s_2)(r_3-1,s_3)}} = \text{Explicitly known ratio of Gamma functions}$$

- We can now algorithmically iterate though conformal fields to three point structure constants

Examples: M(2,5) + M(3,4)





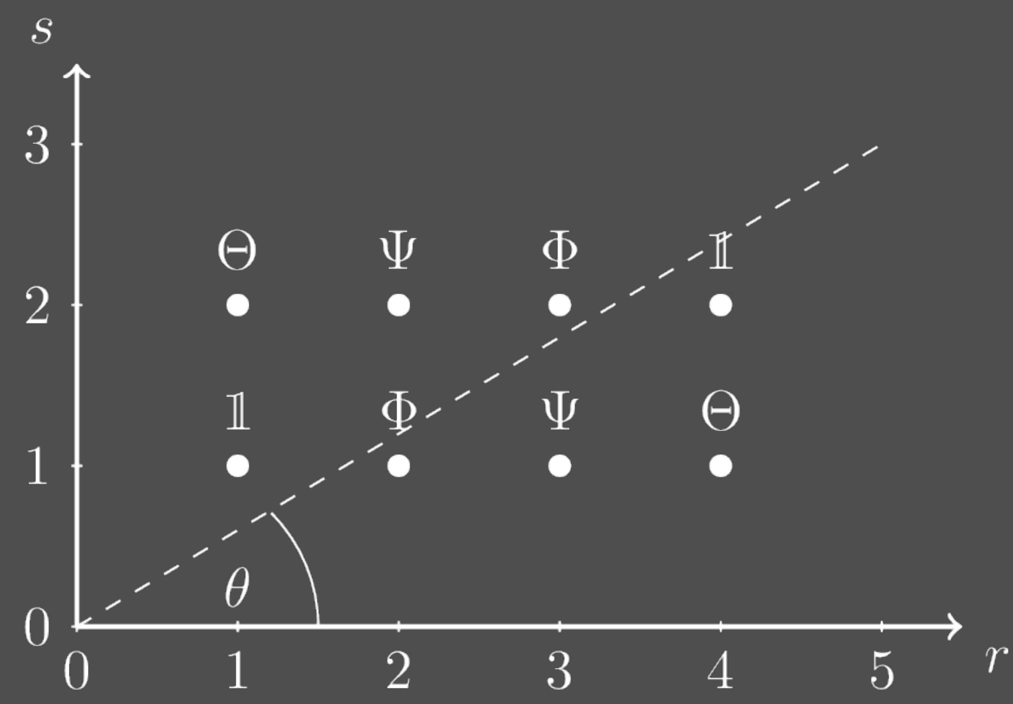
- The Yang-Lee Singularity's only nontrivial three point constant is

$$C^2_{\Phi\Phi\Phi} = -\frac{\Gamma(\frac{6}{5})^2\Gamma(\frac{1}{5})\Gamma(\frac{2}{5})}{\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})^3} \approx -3.65312$$

- The Ising Model's only nontrivial three point constant is

$$C^2_{\sigma\sigma\varepsilon} = \frac{1}{4}$$

Example: M(3,5)



- Only 3-point functions that are not zero are

$$\langle 0|\Phi(z_1)\Phi(z_2)\Psi(z_3)|0\rangle \qquad C^2_{\Phi\Phi\Psi} = -\frac{\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})^3}{\Gamma(\frac{1}{5})\Gamma(\frac{2}{5})\Gamma(\frac{6}{5})^2}$$

$$\langle 0|\Psi(z_1)\Psi(z_2)\Psi(z_3)|0\rangle \qquad C^2_{\Psi\Psi\Psi} = -\frac{\Gamma(-\frac{1}{5})^2\Gamma(\frac{3}{5})^3\Gamma(\frac{4}{5})\Gamma(\frac{7}{5})^2\Gamma(\frac{1}{5})}{\Gamma(-\frac{2}{5})^2\Gamma(\frac{2}{5})^3\Gamma(\frac{6}{5})^4}$$

$$\langle 0|\Phi(z_1)\Phi(z_2)\Theta(z_3)|0\rangle \qquad C^2_{\Phi\Psi\Theta} = \frac{1}{4}$$

Conclusion + Further Directions

- The method described does not compute all structure constants of all minimal modes
- $\mathrm{M}(4{,}5)$ has 9 non-trivial constants and we can compute 6, but are left with 2 equations for the remaining 3
- To compute more one would need to insert more complicated singular vectors
- Another method for computing structure constants using the Coulomb Gas formalism was found by Dotsenko and Fateev
- Apply method to the $N = 1$ super Virasoro minimal models, hopefully with minimal modifications